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Exact solution of the Bogoliubov Hamiltonian for weakly imperfect Bose gas

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Abstract. We show that the pressure of the Bogoliubov weakly imperfect Bose gas (WIBG) can be calculated exactly in the thermodynamic limit. We point out the sufficient and necessary conditions for it not to equate with the pressure of the ideal Bose gas (IBG). We prove that they differ only in that part of the phase diagram where the WIBG has a Bose condensate. We show that in contast to the conventional Bose condensate (e.g. in the IBG) the condensate in the WIBG is due to an effective attraction between bosons in the zero-mode.

1. Introduction and set-up of the problem

A pragmatic procedure for the description of the properties of superfluids, e.g. derivation of the experimentally observed spectra, was initiated in Bogoliubov's classic paper [1] (see also [2]), where he considered a Hamiltonian with *truncated interaction*, giving rise to what is called the Bogoliubov Hamiltonian for a *weakly imperfect Bose gas* (WIBG).

Consider a system of bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^3$ of volume $V = L^3$, with periodic boundary conditions. If $\varphi(x)$ denotes an integrable two-body interaction potential and

$$v(q) = \int_{\mathbb{R}^3} \mathrm{d}^3x\varphi(x)\mathrm{e}^{-\mathrm{i}qx} \qquad q \in \mathbb{R}^3 \tag{1.1}$$

then its second-quantized Hamiltonian acting in the boson Fock space $\mathcal{F}_\Lambda$ can be written as

$$H_\Lambda = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q} v(q) a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \tag{1.2}$$

where all sums run over the set Λ^* defined by

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \alpha = 1, 2, 3, k_\alpha = \frac{2\pi n_\alpha}{L} \text{ and } n_\alpha = 0, \pm 1, \pm 2, \ldots \right\}. \tag{1.3}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ is the one-particle energy spectrum of the free bosons, and $a_k^\# = \{a_k^*, a_k\}$ are the usual boson creation and annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}kx}, k \in \Lambda^*, x \in \Lambda$; $a_k^* \equiv a^*(\psi_k) = \int \Lambda\, \mathrm{d}x\, \psi_k(x) a^*(x)$; $a^\#(x)$ are the basic boson operators in the Fock space $\mathcal{F}_\Lambda$ over $L^2(\Lambda)$. If one supposes that Bose–Einstein condensation, which occurs in the ideal Bose gas for $k = 0$, persists for a weak

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interaction $\varphi(x)$ then, according to Bogoliubov, the most important terms in (1.2) should be those in which at least two operators a_0^*, a_0 appear. We are thus led to consider the following *truncated* Hamiltonian (the Bogoliubov Hamiltonian for WIBG: see [2], part 3.5, equation (3.81)):

$$H_\Lambda^B = T_\Lambda + U_\Lambda^D + U_\Lambda \tag{1.4}$$

where

$$T_\Lambda = \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k \tag{1.5}$$

$$U_\Lambda^D = \frac{v(0)}{V} a_0^* a_0 \sum_{k\in\Lambda^*, k\neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} v(k) a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + \frac{v(0)}{2V} a_0^{*^2} a_0^2 \tag{1.6}$$

$$U_\Lambda = \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} v(k)(a_k^* a_{-k}^* a_0^2 + a_0^{*^2} a_k a_{-k}). \tag{1.7}$$

$H_\Lambda^{BD} \equiv (T_\Lambda + U_\Lambda^D)$ represents the *diagonal* part of the Bogoliubov Hamiltonian H_Λ^B while U_Λ represents the *non-diagonal* part.

Remark 1.1. Below we impose the following assumptions on the two-body interaction potential φ:

(A) $\varphi \in L^1(\mathbb{R}^3)$ (absolute integrability);

(B) $v(k)$ is a real continuous function, satisfying $v(0) > 0$ and $0 \leqslant v(k) = v(-k) \leqslant v(0)$ for $k \in \mathbb{R}^3$.

It is known [3, 4] that under these (and in fact, even weaker) conditions the potential φ is superstable. Hence the grand-canonical partition function associated with the full Hamiltonian (1.2)

$$\Xi_\Lambda(\beta, \mu) = \mathrm{Tr}_{\mathcal{F}_\Lambda}(\mathrm{e}^{-\beta(H_\Lambda - \mu N_\Lambda)}) \tag{1.8}$$

and the finite-volume pressure

$$p_\Lambda[H_\Lambda] \equiv p_\Lambda(\beta, \mu) = \frac{1}{\beta V} \ln \Xi_\Lambda(\beta, \mu) \tag{1.9}$$

are finite for all real chemical potentials μ and all inverse temperatures $\beta > 0$.

However, this is *not true* for the Bogoliubov Hamiltonian (1.4).

Proposition 1.2. [5] Let $\Xi_\Lambda^B(\beta, \mu)$ be the grand-canonical partition function associated with the Hamiltonian (1.4). Then,

(a) the Bogoliubov model of WIBG is stable ($\Xi_\Lambda^B(\beta, \mu) < +\infty$) for $\mu \leqslant 0$ and is unstable (i.e. $\Xi_\Lambda^B(\beta, \mu) = +\infty$) for $\mu > 0$.

(b) For $\mu \leqslant \mu^* = -\frac{1}{2}\varphi(0)$ the pressure

$$p^B(\beta, \mu) = \lim_\Lambda p_\Lambda[H_\Lambda^B] \tag{1.10}$$

coincides with the pressure of the ideal Bose gas (IBG)

$$p^I(\beta, \mu) = \lim_\Lambda p_\Lambda[T_\Lambda]. \tag{1.11}$$

A simple proof of (a) follows from estimates (2.6) and (2.10) below. For the proof of (b) see remark 2.4 and corollary 2.5. Moreover, paper [5] contains the following conjecture.

Conjecture 1.3. The Bogoliubov Hamiltonian H_Λ^B is exactly soluble in the sense that WIBG is thermodynamically equivalent (in the grand-canonical ensemble) to the IBG for all chemical potential $\mu \leqslant 0$. This precisely means that

$$p^B(\beta, \mu \leqslant 0) = p^I(\beta, \mu \leqslant 0). \tag{1.12}$$

The aim of this paper is threefold. First, to show that the phase diagram of the Bogoliubov model (1.4) is less trivial than as expressed by conjecture 1.3 and that it drastically depends on the interaction potential (1.1); secondly, to find necessary and sufficient conditions for the interaction which guarantee that WIBG differs from IBG; and thirdly, to calculate exactly $p^B(\beta,\mu)$ in the domain where it does not coincide with $p^I(\beta,\mu)$.

Our results are organized as follows. In section 2, we show that

$$p^{BD}(\beta,\mu\leqslant 0)=\lim_{\Lambda} p_\Lambda[H_\Lambda^{BD}]=p^I(\beta,\mu\leqslant 0) \tag{1.13}$$

that is, the thermodynamics of the *diagonal* part of the Bogoliubov Hamiltonian and that of the ideal Bose gas coincide, while the Bose condensation is closer to a *generalized condensation*, see [6, 7], which occurs at $k\neq 0$. This means in particular that the thermodynamic *non-equivalence* between the Bogoliubov Hamiltonian and the IBG is due to *non-diagonal* terms of interaction (1.7). We also study conjecture 1.3. We show that for any interaction which satisfies (A) and (B) there is a domain Γ of the phase diagram (plane $Q=(\mu\leqslant 0,\theta=\beta^{-1}\geqslant 0)$) where indeed

$$p^B(\beta,\mu<0)=p^I(\beta,\mu<0). \tag{1.14}$$

We then formulate a sufficient condition on the interaction $v(k)$ to ensure the existence of domain $D_0\subset Q$ where

$$p^B(\beta,\mu)\neq p^I(\beta,\mu). \tag{1.15}$$

In fact we show that this is equivalent to the statement that the system H_Λ^B manifests in this domain a (*non-conventional*) Bose condensation due to *effective attraction* between bosons with $k=0$.

The thermodynamic limit of the pressure (1.10) of the system H_Λ^B in domain $D\supseteq D_0$ defined by

$$p^B(\beta,\mu)\neq p^I(\beta,\mu) \tag{1.16}$$

is studied in section 3. We give an exact formula for $p^B(\beta,\mu)$, demonstrating its relation to the concept of the Bogoliubov approximation as outlined by Ginibre [4]. By corollary we establish that $D=D_0$. In section 4 we study the breaking of the gauge symmetry and the behaviour of the (*non-conventional*) Bose condensate, i.e. the phase diagram of the WIBG. We reserve section 5 for concluding remarks and discussions.

2. Bogoliubov weakly imperfect Bose gas versus ideal Bose gas

2.1. Diagonal part of the Bogoliubov Hamiltonian

The diagonal part of the Bogoliubov Hamiltonian $H_\Lambda^{BD}=(T_\Lambda+U_\Lambda^D)$, as in (1.5) and (1.6), can be rewritten using the occupation-number operators for modes $k\in\Lambda^*$, $n_k=a_k^*a_k$. So, the Hamiltonian $H_\Lambda^{BD}(\mu)\equiv H_\Lambda^{BD}-\mu N_\Lambda$ becomes

$$\begin{aligned}H_\Lambda^{BD}(\mu)=\sum_{k\in\Lambda^*}(\varepsilon_k-\mu)a_k^*a_k+\frac{v(0)}{V}a_0^*a_0\sum_{k\in\Lambda^*,k\neq 0}a_k^*a_k\\ +\frac{1}{2V}\sum_{k\in\Lambda^*,k\neq 0}v(k)a_0^*a_0(a_k^*a_k+a_{-k}^*a_{-k})+\frac{v(0)}{2V}a_0^{*^2}a_0^2\end{aligned} \tag{2.1}$$

where $N_\Lambda = \sum_{k\in\Lambda^*} a_k^* a_k$. If $v(k)$ satisfies (B), then one obviously obtains

$$H_\Lambda^{BD}(\mu) = \sum_{k\in\Lambda^*}(\varepsilon_k - \mu)n_k + \frac{v(0)}{V}n_0 N_\Lambda - \frac{v(0)}{2V}(n_0^2 + n_0) + \frac{1}{V}\sum_{k\in\Lambda^*, k\neq 0} v(k) n_0 n_k \tag{2.2}$$

$$\geqslant T_\Lambda(\mu) \equiv T_\Lambda - \mu N_\Lambda. \tag{2.3}$$

Theorem 2.1. Let $v(k)$ satisfy (A) and (B). Then

(a) for any $\mu \leqslant 0$ and $\beta > 0$ one has

$$p^{BD}(\beta, \mu) \equiv \lim_\Lambda p_\Lambda[H_\Lambda^{BD}] = p^I(\beta, \mu) \tag{2.4}$$

(b) for any $\beta > 0$ one has

$$p^{BD}(\beta, \mu > 0) = +\infty.$$

Proof. (a) By virtue of expression (2.2) and the inequality (2.3) we find that the partition function

$$\Xi_\Lambda^{BD}(\beta, \mu) = \mathrm{Tr}_{\mathcal{F}_\Lambda}\, \mathrm{e}^{-\beta H_\Lambda^{BD}(\mu)} \leqslant \mathrm{Tr}_{\mathcal{F}_\Lambda}\, \mathrm{e}^{-\beta T_\Lambda(\mu)} = \Xi_\Lambda^I(\beta, \mu).$$

Hence, for any $\mu < 0$

$$p_\Lambda[H_\Lambda^{BD}] \leqslant p_\Lambda[T_\Lambda]. \tag{2.5}$$

By (2.2), we can calculate $\mathrm{Tr}_{\mathcal{F}_\Lambda}$ on the basis of occupation-number operators:

$$\Xi_\Lambda^{BD}(\beta, \mu) = \sum_{n_0=0}^{\infty}\left\{\mathrm{e}^{[-\beta(\frac{v(0)}{2V}(n_0^2 - n_0) - \mu n_0)]}\prod_{k\in\Lambda^*, k\neq 0}(1 - \mathrm{e}^{[-\beta(\varepsilon_k - \mu + \frac{[v(0)+v(k)]}{V}n_0)]})^{-1}\right\}$$

which gives the estimate

$$\Xi_\Lambda^{BD}(\beta, \mu) \geqslant \prod_{k\in\Lambda^*, k\neq 0}(1 - \mathrm{e}^{[-\beta(\varepsilon_k - \mu)]})^{-1}.$$

Therefore,

$$p_\Lambda[H_\Lambda^{BD}] \geqslant \tilde{p}_\Lambda^I(\beta, \mu) \equiv \frac{1}{\beta V}\sum_{k\in\Lambda^*, k\neq 0}\ln[(1 - \mathrm{e}^{[-\beta(\varepsilon_k - \mu)]})^{-1}]. \tag{2.6}$$

Note that $\tilde{p}_\Lambda^I(\beta, \mu)$ is the pressure of an ideal Bose gas with *excluded* mode $k = 0$ and $\tilde{p}_\Lambda^I(\beta, \mu) < +\infty$ for $\mu < \inf_{k\neq 0}\varepsilon_k$. Hence, for any $\mu < 0$ one gets

$$\lim_\Lambda \tilde{p}_\Lambda^I(\beta, \mu) = \lim_\Lambda p_\Lambda[T_\Lambda] = p^I(\beta, \mu). \tag{2.7}$$

Therefore, taking the thermodynamic limits as in (2.5)–(2.7) we first solve (2.4) for $\mu < 0$. Then taking limit $\mu \to 0^-$ one solves (2.4) for $\mu = 0$.

(b) is proved directly from estimate (2.6). □

Corollary 2.2. Since functions $\{p_\Lambda^{BD}(\beta, \mu) \equiv p_\Lambda[H_\Lambda^{BD}]\}_{\Lambda\subset\mathbb{R}^d}$ are convex for $\mu \leqslant 0$ and the limit $p^I(\beta, \mu)$ is differentiable for $\mu < 0$, by Griffiths' lemma [8]

$$\lim_\Lambda \partial_\mu p_\Lambda^{BD}(\beta, \mu) = \partial_\mu p^I(\beta, \mu)$$

i.e. the particle density of the system (2.1) coincides with that of the IBG:

$$\rho^{BD}(\beta, \mu) \equiv \lim_\Lambda \left\langle \frac{N}{V}\right\rangle_{H_\Lambda^{BD}}(\beta, \mu) = \partial_\mu p^I(\beta, \mu) \equiv \rho^I(\beta, \mu). \tag{2.8}$$

Here $\langle - \rangle_{H_\Lambda}(\beta, \mu)$ corresponds to the grand-canonical Gibbs state for Hamiltonian H_Λ. Taking in (2.8) the limit $\mu \to 0^-$ we extend this equality to $\mu \in (-\infty, 0]$.

From (2.4) and (2.8) we see that the diagonal part of the Bogoliubov Hamiltonian H_Λ^{BD} is thermodynamically equivalent to T_Λ. A *generalized Bose condensation* in the system (2.1) coincides with that for the ideal Bose gas with excluded mode $k = 0$, [6, 7]. Below we show that it is the *non-diagonal* interaction (1.7) that makes the essential difference between WIBG and IBG.

2.2. Domain Γ: $p^B(\beta, \mu) = p^I(\beta, \mu)$

As with IBG, the Bogoliubov WIBG exists only for $\mu \leqslant 0$ (see proposition 1.2). In fact we can go further (cf [5]).

Lemma 2.3. For any $\mu \leqslant 0$, one has

$$p^I(\beta, \mu) \leqslant p^B(\beta, \mu). \tag{2.9}$$

Proof. By the Bogoliubov inequality (see e.g. [3, 9]), one knows that for any $\mu \leqslant 0$,

$$\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^B} \leqslant p_\Lambda[H_\Lambda^{BD}] - p_\Lambda[H_\Lambda^B] \leqslant \frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^{BD}}. \tag{2.10}$$

Since $\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^{BD}} = 0$, combining (2.6), (2.7) and (2.10) we obtain (2.9) in the thermodynamic limit by the continuous extension $\mu \to 0^-$ of $p^I(\beta, \mu)$ for $\mu \leqslant 0$. □

Remark 2.4. Let $v(k)$ satisfy (A) and (B). Then regrouping terms in (1.6), (1.7) one gets

$$H_\Lambda^B = \tilde{H}_\Lambda + \frac{1}{2V}\sum_{k\in\Lambda^*, k\neq 0} v(k)(a_0^* a_k + a_{-k}^* a_0)^*(a_0^* a_k + a_{-k}^* a_0) \geqslant \tilde{H}_\Lambda \tag{2.11}$$

where

$$\tilde{H}_\Lambda = \sum_{k\in\Lambda^*, k\neq 0}\left(\varepsilon_k - \frac{v(k)}{2V} + \frac{v(0)}{V}n_0\right)n_k + \frac{v(0)}{2V}n_0^2 - \frac{1}{2}\varphi(0)n_0. \tag{2.12}$$

Hence, by (2.11) and (2.12) we obtain in the thermodynamic limit for $\mu \leqslant 0$

$$p^B(\beta, \mu) \leqslant \lim_\Lambda p_\Lambda[\tilde{H}_\Lambda] = \sup_{\rho_0 \geqslant 0} G(\beta, \mu; \rho_0). \tag{2.13}$$

Here

$$G(\beta, \mu; \rho_0) \equiv \left[-\frac{v(0)}{2}\rho_0^2 + (\mu + \tfrac{1}{2}\varphi(0))\rho_0 + p^I(\beta, \mu - v(0)\rho_0)\right]. \tag{2.14}$$

Corollary 2.5. [5] If $\mu \leqslant -\frac{1}{2}\varphi(0)$, then $\sup_{\rho_0 \geqslant 0} G(\beta, \mu; \rho_0) = p^I(\beta, \mu)$. Therefore, by lemma 2.3 and inequality (2.13) we get

$$p^B(\beta, \mu) = p^I(\beta, \mu) \text{ for } \Gamma_{\mu_*} = \{\theta \geqslant 0, \mu \leqslant -\tfrac{1}{2}\varphi(0) \equiv \mu_*\}. \tag{2.15}$$

The next statement extends the domain Γ_{μ_*}, (see figure 1) of proposition 1.2.

Theorem 2.6. Let $v(k)$ satisfy (A) and (B) and let

$$h(z, \beta) \equiv z + \frac{v(0)}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k\, (\mathrm{e}^{[\beta(\varepsilon_k + z)]} - 1)^{-1}. \tag{2.16}$$

Then we have

$$p^B(\beta, \mu) = p^I(\beta, \mu) \qquad \text{for } (\theta, \mu) \in \Gamma \tag{2.17}$$

where

$$\Gamma = \left\{(\theta, \mu) : \tfrac{1}{2}\varphi(0) \leqslant \inf_{z \geqslant -\mu} h(z, \beta)\right\} \subset Q. \tag{2.18}$$

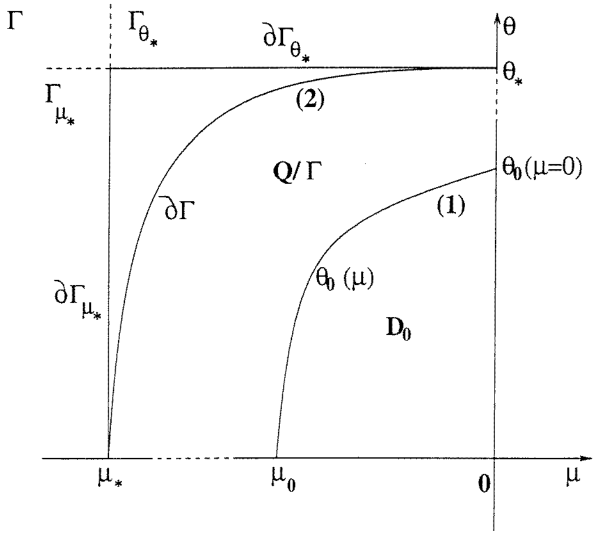

Figure 1. (1) Phase diagram of the Bogoliubov WIBG. (2) Estimation of the phase diagram given in [5] and in section 2. Note that deviation of the boundary $\partial\Gamma$ from the straight lines $\partial\Gamma_{\mu_*}$, $\partial\Gamma_{\theta_*}$ is exaggerated in the vinicity of (μ_*, θ_*). Numerical estimates give $\mu_*/\mu_0 \sim 10^3$, $\theta_*/\theta_0 \sim 10$.

Proof. By virtue of (2.9), (2.13) and (2.14), the equality (2.17) is ensured by

$$\sup_{\rho_0 \geqslant 0} G(\beta, \mu; \rho_0) = G(\beta, \mu; \rho_0 = 0). \tag{2.19}$$

If $\partial_{\rho_0} G(\beta, \mu; \rho_0) \leqslant 0$ or equivalently $\frac{1}{2}\varphi(0) \leqslant h(v(0)\rho_0 - \mu, \beta)$ for $\rho_0 \geqslant 0$, then sufficient condition (2.18) guarantees (2.19) and hence (2.17). □

Corollary 2.7. Since $h(z, \beta)$ is a convex function of $z \geqslant 0$ and $h(z, \beta) \geqslant z$, then by (2.16) we get

$$\lambda(\theta) \leqslant \inf_{z \geqslant -\mu} h(z, \beta) \tag{2.20}$$

where

$$\lambda(\theta) \equiv \inf_{z \geqslant 0} h(z, \beta). \tag{2.21}$$

Therefore, by (2.20) we get a sufficient condition independent of $\mu \leqslant 0$ (high-temperature domain, see figure 1):

$$\Gamma_{\theta_*} = \{(\theta, \mu \leqslant 0) : \tfrac{1}{2}\varphi(0) \equiv \lambda(\theta_*) \leqslant \lambda(\theta)\} \tag{2.22}$$

which ensures (2.17).

Remark 2.8. Note that the inequality $h(z, \beta) \geqslant z$ and (2.18) implies (2.15) for $-\mu \geqslant \frac{1}{2}\varphi(0)$, i.e. $\Gamma_{\mu_*} \subset \Gamma$. On the other hand, (2.18) for $\mu = 0$ implies (2.22), i.e. $\Gamma_{\theta_*} \subset \Gamma$, see figure 1.

Remark 2.9. Since $\partial_{v(0)}\lambda(\theta) \geqslant 0$, one can always ensure (2.22) for a fixed temperature θ, by increasing $v(0)$ without changing $\varphi(0)$.

Remark 2.10. Note that $p^I(\beta = +\infty, \mu) = 0$ and that $\lambda(\theta = 0) = 0$. Therefore, at zero temperature the sufficient condition (2.18) reduces to (2.15), see figure 1. In fact this part of Γ is known from [5]. Theorem 2.6 shows that conjecture 1.3 can be extended at least to the domain Γ (2.18).

Below we show that this conjecture is *not* valid in the complement $Q\backslash\Gamma$, see figure 1.

2.3. Domain D: $p^B(\beta, \mu) \neq p^I(\beta, \mu)$

Let $\mathcal{H}_{0\Lambda} \subset L^2(\Lambda)$ be the one-dimensional subspace generated by $\psi_{k=0}$ (see section 1). Then the Fock space $\mathcal{F}_\Lambda$ is isomorphic to the tensor product $\mathcal{F}_\Lambda \approx \mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_\Lambda$ where $\mathcal{F}_{0\Lambda}$ and $\mathcal{F}'_\Lambda$ are the boson Fock spaces constructed out of $\mathcal{H}_{0\Lambda}$ and of its orthogonal complement $\mathcal{H}^\perp_{0\Lambda}$ respectively. For any complex $c \in \mathbb{C}$, we can define in $\mathcal{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = \mathrm{e}^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} (\sqrt{V}c)^k (a_0^*)^k \Omega_0 \tag{2.23}$$

where Ω_0 is the vacuum of $\mathcal{F}_\Lambda$. Then $a_0\psi_{0\Lambda}(c) = \sqrt{V}c\psi_{0\Lambda}(c)$.

Definition 2.11. [4] The Bogoliubov approximation for a Hamiltonian $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ in $\mathcal{F}_\Lambda$ is the operator $H_\Lambda(c^\#, \mu)$ defined in $\mathcal{F}'_\Lambda$ by its quadratic form

$$(\psi'_1, H_\Lambda(c^\#, \mu)\psi'_2)_{\mathcal{F}'_\Lambda} = (\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\psi_{0\Lambda}(c) \otimes \psi'_2)_{\mathcal{F}_\Lambda} \tag{2.24}$$

for $\psi_{0\Lambda}(c) \otimes \psi'_{1,2}$ in the form-domain of $H_\Lambda(\mu)$, where $c^\# = (c, \bar{c})$.

This formulation of the Bogoliubov approximation [1, 2] provides an estimate for the pressure $p_\Lambda[H^B_\Lambda]$ from below which allows us to refine (2.9).

Proposition 2.12. [5] For any $(\theta, \mu) \in Q$ one has

$$\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu; c^\#) \leqslant p_\Lambda[H^B_\Lambda] \tag{2.25}$$

where

$$\tilde{p}^B_\Lambda(\beta, \mu; c^\#) \equiv \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}'_\Lambda} \mathrm{e}^{-\beta H^B_\Lambda(c^\#, \mu)}. \tag{2.26}$$

Remark 2.13. By definition 2.11 we get from (1.4)–(1.7) that

$$\begin{aligned} H^B_\Lambda(c^\#, \mu) = &\sum_{k\in\Lambda^*, k\neq 0} [\varepsilon_k - \mu + v(0)|c|^2] a^*_k a_k + \tfrac{1}{2} \sum_{k\in\Lambda^*, k\neq 0} v(k)|c|^2 [a^*_k a_k + a^*_{-k} a_{-k}] \\ &+ \tfrac{1}{2} \sum_{k\in\Lambda^*, k\neq 0} v(k)[c^2 a^*_k a^*_{-k} + \bar{c}^2 a_k a_{-k}] - \mu|c|^2 V + \tfrac{1}{2} v(0)|c|^4 V. \end{aligned} \tag{2.27}$$

Therefore, after diagonalization one can calculate (2.26) in the explicit form:

$$\begin{aligned} &\tilde{p}^B_\Lambda(\beta, \mu; c^\#) = \xi_\Lambda(\beta, \mu; x) + \eta_\Lambda(\mu; x) \\ &\xi_\Lambda(\beta, \mu; x) = \frac{1}{\beta V} \sum_{k\in\Lambda^*, k\neq 0} \ln(1 - \mathrm{e}^{-\beta E_k})^{-1} \\ &\eta_\Lambda(\mu; x) = -\frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} (E_k - f_k) + \mu x - \tfrac{1}{2} v(0) x^2 \end{aligned} \tag{2.28}$$

where E_k and f_k are functions of $x = |c|^2 \geqslant 0$ and $\mu \leqslant 0$:

$$\begin{aligned} &f_k = \varepsilon_k - \mu + x[v(0) + v(k)] \\ &h_k = x v(k) \\ &E_k = \sqrt{f_k^2 - h_k^2}. \end{aligned} \tag{2.29}$$

Now the strategy of localization of the domain D_0 (see figure 1) becomes clear: by virtue of (2.9) and (2.25) one comes to describe the set of $(\theta, \mu) \in Q$ such that

$$p^I(\beta, \mu) < \lim_{\Lambda} \left[\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu; c^\#) \right]. \tag{2.30}$$

Proposition 2.14. [5] Let $v(k)$ satisfy (A), (B) and

$$v(0) \geqslant \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3k \, \frac{[v(k)]^2}{\varepsilon_k}. \tag{2.31}$$

Then, cf (2.6),

$$\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu; c^\#) = \tilde{p}^B_\Lambda(\beta, \mu; 0) = \tilde{p}^I_\Lambda(\beta, \mu).$$

Therefore, in the thermodynamic limit (see (2.7)) we get

$$\lim_{\Lambda} \left[\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu; c^\#) \right] = p^I(\beta, \mu). \tag{2.32}$$

Lemma 2.15. Let $v(k)$ satisfy (A), (B) and the condition (C):

$$v(0) < \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3k \, \frac{[v(k)]^2}{\varepsilon_k}. \tag{2.33}$$

Then, there is $\mu_0 < 0$ such that

$$\lim_{\Lambda} \left(\sup_{x\geqslant 0} \eta_\Lambda(\mu; x) \right) = \eta(\mu; \overline{x}(\mu) > 0) > 0 \text{ for } \mu \in (\mu_0, 0]. \tag{2.34}$$

Proof. By the explicit formulae (2.28) and (2.29) we readily find that for $\mu \leqslant 0$:

(a) $\eta_\Lambda(\mu; x = 0) = 0$ and $\eta_\Lambda(\mu; x) \leqslant \text{constant} - \frac{1}{2}v(0)x^2$

(b) $\partial_x \eta_\Lambda(\mu; x = 0) = \mu$ and

$$\partial_x^2 \eta_\Lambda(\mu; x = 0) = \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} - v(0).$$

Since

$$\lim_{\Lambda} \frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3k \, \frac{[v(k)]^2}{(\varepsilon_k - \mu)}$$

the condition (2.33) implies the existence of $\tilde{\mu} < 0$ such that

$$\lim_{\Lambda} \partial_x^2 \eta_\Lambda(\mu > \tilde{\mu}; x = 0) > 0.$$

By virtue of (a), (b), and $\lim_\Lambda \partial_x \eta_\Lambda(\mu = 0; x = 0) = 0$ this means that

$$\lim_{\Lambda} \left(\sup_{x\geqslant 0} \eta_\Lambda(\mu = 0; x) \right) = \eta(\mu = 0; \overline{x}(\mu = 0) > 0) > 0. \tag{2.35}$$

Therefore, by continuity of (2.35) on the interval $(\tilde{\mu}, 0]$ we get the existence of $\mu_0 : \tilde{\mu} \leqslant \mu_0 < 0$, such that one has (2.34). □

Theorem 2.16. Let $v(k)$ satisfy (A)–(C). Then, for any $\mu \in (\mu_0, 0]$, there is $\theta_0(\mu) > 0$ such that one has (see figure 1):

$$p^I(\beta, \mu) < p^B(\beta, \mu) \text{ in } D_0 \equiv \{(\theta, \mu) : \mu_0 < \mu \leqslant 0, 0 \leqslant \theta < \theta_0(\mu)\} \tag{2.36}$$

where μ_0 is defined by lemma 2.15. In fact the domain D_0 coincides with

$$D_0 = \left\{ (\theta, \mu) : \lim_{\Lambda} \sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu; c^\#) > p^I(\beta, \mu) \right\}.$$

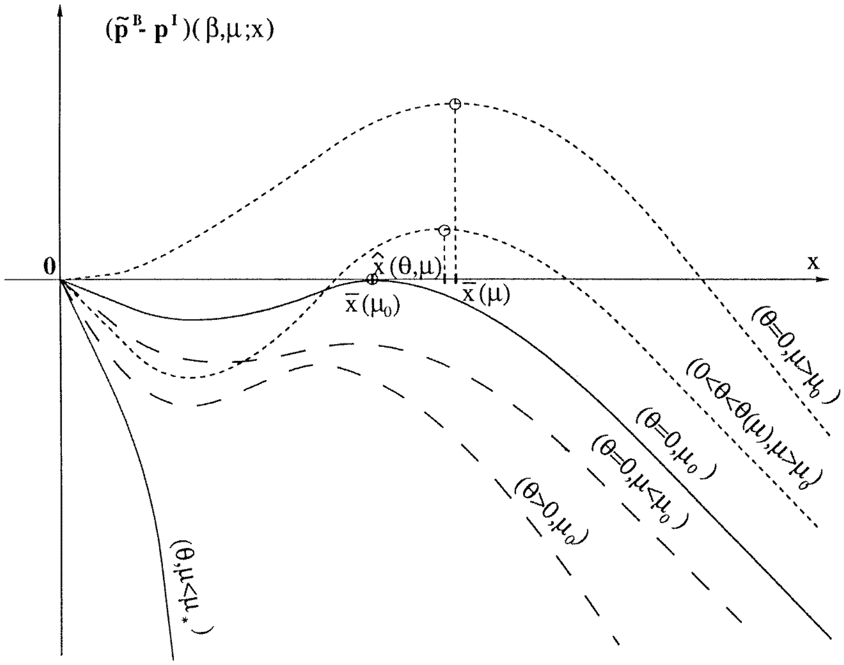

Figure 2. Illustration of the Bogoliubov approximation variational problem: the behaviour of the difference between the trial pressure $\tilde{p}^B(\beta, \mu; x)$ for the WIBG and the IBG pressure $p^I(\beta, \mu)$ as a function of the variational parameter $x = |c|^2$ for different values of (θ, μ). Non-trivial suprema are indicated by empty circles.

Proof. First we note that by (2.28) and (2.29) one has $\xi_\Lambda(\beta, \mu; x = 0) = \tilde{p}^I_\Lambda(\beta, \mu)$ and that in addition:

$$\begin{aligned}
&\text{(i) } \partial_x \xi_\Lambda(\beta, \mu; x) \leqslant 0 \qquad \text{and} \qquad \lim_{x\to+\infty} \xi_\Lambda(\beta, \mu; x) = 0 \qquad \text{for any } \Lambda \\
&\text{(ii) } \partial_\theta \xi_\Lambda(\beta, \mu; x) \geqslant 0 \qquad \text{and} \qquad \lim_{\theta\to 0} \xi_\Lambda(\beta, \mu; x) = 0 \qquad \text{for any } \Lambda.
\end{aligned} \tag{2.37}$$

Next, by lemma 2.15 for $\mu = \mu_0 < 0$ we have

$$\lim_\Lambda \left(\sup_{x\geqslant 0} \eta_\Lambda(\mu_0; x) \right) = \eta(\mu_0; 0) = \eta(\mu_0; \bar{x}(\mu_0) > 0) = 0. \tag{2.38}$$

Hence, according to (2.37) and (2.38) one obtains:

$$\begin{aligned}
\text{(iii) } \theta > 0 : \lim_\Lambda \left[\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu_0; c^\#) \right] &= \sup_{x\geqslant 0} [\xi(\beta, \mu_0; x) + \eta(\mu_0; x)] \\
&= \tilde{p}^B(\beta, \mu_0; c^\# = 0) = p^I(\beta, \mu)
\end{aligned} \tag{2.39}$$

and by (2.37), (ii) and (2.38), we obtain:

$$\begin{aligned}
\text{(iv) } \theta = 0 : \lim_\Lambda \left[\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta = \infty, \mu_0; c^\#) \right] &= \tilde{p}^B(\beta = \infty, \mu_0; c^\# = 0) \\
&= \tilde{p}^B(\beta = \infty, \mu_0; c^\#)|_{|c|^2 = \bar{x}(\mu_0) > 0} = 0
\end{aligned}$$

see figures 1 and 2.

Now by (2.28), (2.37) and lemma 2.15 one obtains that for $\mu_0 < \mu \leqslant 0$

$$\lim_\Lambda \left[\sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta, \mu > \mu_0; c^\#) \right] \geqslant \eta(\mu > \mu_0; \bar{x}(\mu) > 0) > 0. \tag{2.40}$$

Since by (2.37) (ii) the pressure $p^I(\beta, \mu \leqslant 0)$ is monotonously decreasing for $\theta \searrow 0$, there is a temperature $\theta_0(\mu)$such that for $\theta < \theta_0(\mu > \mu_0)$ we get from (2.40)

$$p^I(\beta > \beta_0(\mu), \mu > \mu_0) < \eta(\mu > \mu_0; \overline{x}(\mu) > 0) < \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}^B_\Lambda(\beta > \beta_0(\mu), \mu > \mu_0; c^\#) \right]. \tag{2.41}$$

Then (2.25) and (2.41) imply (2.36) for $(\theta, \mu) \in D_0$ which is equivalent to (2.30). □

Corollary 2.17. Let

$$D \equiv \{(\theta, \mu) : p^B(\beta, \mu) > p^I(\beta, \mu)\}. \tag{2.42}$$

Then by (2.25) and (2.36) one obviously gets

$$D \supseteq D_0 = \{(\theta, \mu) : \mu_0 < \mu \leqslant 0, 0 \leqslant \theta < \theta_0(\mu)\}.$$

Here $\mu_0 < 0$ is defined in lemma 2.15 and $\theta_0(\mu)$ in theorem 2.16.

Remark 2.18. The condition (C) defined by (2.33) is sufficient to guarantee that $\mu_0 < 0$, i.e. $D \supseteq D_0 \neq \{\emptyset\}$. On the other hand, the contrary condition (2.31) implies only the triviality (2.32) of the lower bound (2.25) for $p^B(\beta, \mu)$ but not $D = \{\emptyset\}$, see lemma 2.3 and (2.30).

Therefore, for the moment we do not know whether condition (C) is *necessary* for $D \neq \{\emptyset\}$. We postpone seeking the answer to this question until section 3. Below we remark on a relation between conditions (2.31) and (2.33) (which result from a rather restricted analysis of convexity and monotonicity of the $\tilde{p}^B_\Lambda(\beta, \mu; c^\#)$ in the vicinity of $x = 0$) and the condition (2.15), which gives triviality to the upper bound (2.13) for $p^B(\beta, \mu)$ for all temperatures (see figure 1).

Remark 2.19. Let $v(k)$ satisfy (A)–(C). Then there is $\tilde{\mu} < 0$ such that for $\mu \leqslant \tilde{\mu}$ one has

$$v(0) \geqslant \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} \, \mathrm{d}^3k \tag{2.43}$$

and in consequence $\partial_x^2 \eta(\mu \leqslant \tilde{\mu}; x = 0) \leqslant 0$ (see the proof of lemma 2.15). One can represent the inequality (2.43) as

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3} v(k) \left\{ \frac{v(k)}{2(\varepsilon_k - \mu)} - \frac{v(0)}{\varphi(0)} \right\} \leqslant 0. \tag{2.44}$$

Since by (B) and by $\mu \leqslant 0$ we have

$$\frac{v(k)}{2(\varepsilon_k - \mu)} \leqslant \frac{v(0)}{(-2\mu)}$$

the condition $\mu < -\frac{1}{2}\varphi(0) \equiv \mu_*$ (2.15) implies (2.44), i.e. $\mu_* \leqslant \tilde{\mu}$, see figures 1 and 2. Therefore, a local convexity condition (2.43) for $\eta_\Lambda(\mu; x)$ is intimately related to the condition ensuring $p^B(\beta, \mu) = p^I(\beta, \mu)$. In particular, notice that for the condition (2.31) the inequality (2.43) is valid for any $\mu \leqslant 0$.

We conclude this section with a simple and important theorem for characterization of domain D (cf (2.42)).

Theorem 2.20. Let

$$\rho^B_0(\beta, \mu) \equiv \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H^B_\Lambda} (\beta, \mu) \tag{2.45}$$

be the density of the Bose condensate in the Bogoliubov WIBG (1.4). Then

$$D = \{(\theta, \mu) \in Q : \rho^B_0(\beta, \mu) > 0\}. \tag{2.46}$$

Proof. Put

$$\hat{H}_\Lambda^B \equiv H_\Lambda^B + \tfrac{1}{2}\varphi(0)a_0^* a_0. \tag{2.47}$$

Then by remark 2.4 we get

$$\lim_\Lambda p_\Lambda\left[\hat{H}_\Lambda^B\right] \leqslant \sup_{\rho_0 \geqslant 0}\{G(\beta,\mu;\rho_0) - \tfrac{1}{2}\varphi(0)\rho_0\} = p^I(\beta,\mu). \tag{2.48}$$

By the Bogoliubov inequality for H_Λ^B and $\hat{H}_\Lambda^B$ one has

$$p_\Lambda[H_\Lambda^B] - \frac{\varphi(0)}{2}\left\langle \frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B} \leqslant p_\Lambda[\hat{H}_\Lambda^B]. \tag{2.49}$$

Hence, by virtue of (2.9), (2.48) and (2.49) we get in the thermodynamic limit that

$$p^I(\beta,\mu) - \frac{\varphi(0)}{2}\rho_0^B(\beta,\mu) \leqslant p^B(\beta,\mu) - \frac{\varphi(0)}{2}\rho_0^B(\beta,\mu) \leqslant p^I(\beta,\mu).$$

Therefore, $p^B(\beta,\mu) = p^I(\beta,\mu)$ if and only if $\rho_0^B(\beta,\mu) = 0$, which gives (2.46). □

Remark 2.21. The observation that $p^B(\beta,\mu) \neq p^I(\beta,\mu)$ only when $\rho_0^B(\beta,\mu) \neq 0$ is very similar to what is known since Bogoliubov theory of superfluidity [1, 2]. An essential difference is that in the Bogoliubov theory the gapless spectrum occurs for a positive chemical potential $\mu = v(0)\rho_0^B$ where the system corresponding to the Bogoliubov Hamiltonian for WIBG is unstable. For further discussion see [5, 10, 11] and section 5.

3. Exactness of the Bogoliubov approximation

Since the pressure $p^B(\beta,\mu) \neq p^I(\beta,\mu)$ only in domain D, where the Bose condensate $\rho_0^B(\beta,\mu) > 0$, the aim of this section is to identify $p^B(\beta,\mu)$ in this domain. Below we shall show that

$$p^B(\beta,\mu) = \lim_\Lambda\left[\sup_{c\in\mathbb{C}} \tilde{p}_\Lambda^B(\beta,\mu;c^\#)\right] = \sup_{c\in\mathbb{C}} \tilde{p}^B(\beta,\mu;c^\#) \tag{3.1}$$

and that in fact (cf (2.36) and (2.42)) one has

$$D = D_0. \tag{3.2}$$

Therefore, the condition (C) (2.33) is *sufficient* and *necessary* for $D \neq \{\emptyset\}$ (see remark 2.18). By definition of $\tilde{p}^B(\beta,\mu;c^\#)$, (see (2.25)–(2.28)), the statement (3.1) means that the Bogoliubov approximation for the WIBG is *exact*. Since $\tilde{p}_\Lambda^B(\beta,\mu;c^\#)$ is known explicitly, the statement (3.1) gives the *exact solution* of thermodynamics of this model.

In section 2 we showed that it is the non-diagonal part U_Λ (1.7) of the Bogoliubov Hamiltonian (1.4) that ensures that $p^B(\beta,\mu) \neq p^I(\beta,\mu)$ in domain $D \neq \{\emptyset\}$. The interaction U_Λ is known to be *effectively attractive* [5], and given condition (C), it prevails over the term of direct repulsive interaction between bosons for the mode $k = 0$ (see (1.6)) [12]. Therefore, to prove (3.1) we use the approximation Hamiltonian method originally invented for quantum systems with attractive interactions (see e.g. [9]).

Remark 3.1. This method was adapted by Ginibre [4] to prove the exactness of the Bogoliubov approximation for a non-ideal Bose gas (1.2) with superstable interaction, which is the case of $v(q)$ satisfying (B). But after truncation of (1.2) the Hamiltonian H_Λ^B (1.4) for WIBG is not superstable. By proposition 1.2, the system (1.4) is unstable for $\mu > 0$. Below we follow the approximation Hamiltonian method as used by Ginibre, improved for the WIBG.

Since in the approximating Hamiltonian $H_\Lambda^B(c^\#, \mu)$ (2.27) the gauge symmetry is broken, we introduce

$$\begin{aligned} H_\Lambda^B(\nu^\#) &= H_\Lambda^B - \sqrt{V}(\overline{\nu} a_0 + \nu a_0^*) \\ H_\Lambda^B(\mu, \nu^\#) &= H_\Lambda^B(\nu^\#) - \mu N_\Lambda \end{aligned} \tag{3.3}$$

with sources $\nu \in \mathbb{C}$ breaking the symmetry of H_Λ^B, here $\nu^\# = (\nu, \overline{\nu})$. Then by proposition 2.12 and the Bogoliubov inequality for $H_\Lambda^B(\mu, \nu^\#)$ and $H_\Lambda^B\left(c^\#, \mu, \nu^\#\right)$ one gets:

$$\begin{aligned} 0 \leqslant \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) &\equiv p_\Lambda[H_\Lambda^B(\nu^\#)] - \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \\ &\leqslant \frac{1}{V}\langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#)\rangle_{H_\Lambda^B(\nu^\#)}. \end{aligned} \tag{3.4}$$

Let $A \equiv a_0 - \sqrt{V}c$, $A^* \equiv a_0^* - \sqrt{V}\overline{c}$. Then a Taylor expansion around $a_0^\#$ gives:

$$\begin{aligned} H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) &= -A^*[a_0, H_\Lambda^B(\mu, \nu^\#)] + \text{h.c.} + \tfrac{1}{2}A^{*^2}[a_0, [a_0, H_\Lambda^B(\mu, \nu^\#)]] \\ &\quad + \text{h.c.} + A^*[a_0, [H_\Lambda^B(\mu, \nu^\#), a_0^*]]A - \tfrac{1}{2}A^{*^2}[a_0, [a_0, [H_\Lambda^B(\mu, \nu^\#), a_0^*]]]A \\ &\quad + \text{h.c.} + \tfrac{1}{4}A^{*^2}[a_0, [a_0, [[H_\Lambda^B(\mu, \nu^\#), a_0^*]a_0^*]]]A^2. \end{aligned} \tag{3.5}$$

Remark 3.2. Explicit calculations show that the third and the fourth order terms in (3.5) are bounded from above:

$$\begin{aligned} -\frac{v(0)}{\sqrt{V}}(\overline{c}A^*AA + cA^*A^*A) - \frac{v(0)}{2V}A^{*^2}A^2 &= 2v(0)|c|^2A^*A \\ -\frac{v(0)}{2V}(A^2 + 2\sqrt{V}cA)^*(A^2 + 2\sqrt{V}cA) &\leqslant 2v(0)|c|^2A^*A. \end{aligned} \tag{3.6}$$

Remark 3.3. After some algebra, the terms of the first and the second order in (3.5) can be combined in

$$\begin{aligned} &-\tfrac{1}{2}[A^*A, [H_\Lambda^B(\mu, \nu^\#), A^*A]] + 2A^*[A, [H_\Lambda^B(\mu, \nu^\#), A^*]]A \\ &\qquad -\tfrac{3}{2}A^*[A, H_\Lambda^B(\mu, \nu^\#)] - \tfrac{3}{2}[H_\Lambda^B(\mu, \nu^\#), A^*]A. \end{aligned} \tag{3.7}$$

Lemma 3.4. One has the following inequality:

$$\langle[A^*A, [H_\Lambda^B(\mu, \nu^\#), A^*A]]\rangle_{H_\Lambda^B(\nu^\#)} \geqslant 0. \tag{3.8}$$

Proof. Denote by $(.,.)_{H_\Lambda}$ the positive semidefinite scalar product with respect to a Hamiltonian H_Λ (see e.g. [13]):

$$(X, Y)_{H_\Lambda} \equiv \frac{1}{\beta\,\Xi_\Lambda(\beta, \mu)}\int_0^\beta \mathrm{d}\tau\ \mathrm{Tr}_{\mathcal{F}_\Lambda}(\mathrm{e}^{-(\beta-\tau)H_\Lambda(\mu)}X^*\mathrm{e}^{-\tau H_\Lambda(\mu)}Y). \tag{3.9}$$

Then $(\mathbf{1}, Y)_{H_\Lambda} = \langle Y\rangle_{H_\Lambda}$ and

$$\beta([X, H_\Lambda(\mu)], [X, H_\Lambda(\mu)])_{H_\Lambda} = \langle[X, [H_\Lambda(\mu), X^*]]\rangle_{H_\Lambda}. \tag{3.10}$$

Applying (3.10) to $H_\Lambda(\mu) = H_\Lambda^B(\mu, \nu^\#)$ and $X = A^*A$ one gets (3.8). □

Lemma 3.5. One has the following estimate:

$$\begin{aligned} -2\langle A^*[A, H_\Lambda^B(\mu, \nu^\#)]\rangle_{H_\Lambda^B(\nu^\#)} &\leqslant \langle[A^*, [H_\Lambda^B(\mu, \nu^\#), A]]\rangle_{H_\Lambda^B(\nu^\#)} \\ &\quad + \langle[A^*, [H_\Lambda^B(\mu, \nu^\#), A]]^*\rangle_{H_\Lambda^B(\nu^\#)} + 2\beta^{-1}\langle\{A, A^*\}\rangle_{H_\Lambda^B(\nu^\#)} \end{aligned} \tag{3.11}$$

where $\{X, Y\} \equiv XY + YX$.

Proof. By the spectral decomposition of the Hamiltonian ($H_\Lambda^B(\mu,\nu^\#)\psi_n = E_n\psi_n$) one gets

$$\langle\{A^*,[H_\Lambda^B(\mu,\nu^\#),A]\}\rangle_{H_\Lambda^B(\nu^\#)} = \frac{1}{\Xi_\Lambda^B(\beta,\mu,\nu^\#)}\sum_{m,n}|(\psi_m,A\psi_n)|^2(E_m-E_n)(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m}). \tag{3.12}$$

Since

$$\tfrac{1}{2}(\mathrm{e}^x+\mathrm{e}^y)-\tfrac{1}{2}|\mathrm{e}^x-\mathrm{e}^y| \leqslant \frac{\mathrm{e}^x-\mathrm{e}^y}{x-y} \leqslant \tfrac{1}{2}(\mathrm{e}^x+\mathrm{e}^y) \tag{3.13}$$

one gets

$$\begin{aligned}\beta(E_m-E_n)(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m}) &\leqslant 2(\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_m})+\beta(E_m-E_n)|\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_m}|\\ &\leqslant 2(\mathrm{e}^{-\beta E_n}+\mathrm{e}^{-\beta E_m})+\beta(E_m-E_n)(\mathrm{e}^{-\beta E_n}-\mathrm{e}^{-\beta E_m}).\end{aligned} \tag{3.14}$$

Inserting the estimate (3.14) into (3.12) we obtain

$$\langle\{A^*,[H_\Lambda^B(\mu,\nu^\#),A]\}\rangle_{H_\Lambda^B(\nu^\#)} \leqslant 2\beta^{-1}\langle AA^*+A^*A\rangle_{H_\Lambda^B(\nu^\#)} + \langle[A^*,[H_\Lambda^B(\mu,\nu^\#),A]]\rangle_{H_\Lambda^B(\nu^\#)}. \tag{3.15}$$

Note that

$$\begin{aligned}-2\langle A^*[A,H_\Lambda^B(\mu,\nu^\#)]\rangle_{H_\Lambda^B(\nu^\#)} &= \langle[A^*,[H_\Lambda^B(\mu,\nu^\#),A]]\rangle_{H_\Lambda^B(\nu^\#)}\\ &\quad+\langle\{A^*,[H_\Lambda^B(\mu,\nu^\#),A]\}\rangle_{H_\Lambda^B(\nu^\#)}.\end{aligned} \tag{3.16}$$

Then combining (3.15) and (3.16) one gets (3.11). □

Corollary 3.6. Since

$$\langle A^*[A,H_\Lambda^B(\mu,\nu^\#)]\rangle_{H_\Lambda^B(\nu^\#)} = \langle[H_\Lambda^B(\mu,\nu^\#),A^*]A\rangle_{H_\Lambda^B(\nu^\#)}$$

by the estimate (3.11) the mean value of the last two terms of (3.7) is bounded from above:

$$\begin{aligned}-3\langle A^*[A,H_\Lambda^B(\mu,\nu^\#)]\rangle_{H_\Lambda^B(\nu^\#)} &\leqslant \tfrac{3}{2}\langle[A^*,[H_\Lambda^B(\mu,\nu^\#),A]]+\text{h.c.}\rangle_{H_\Lambda^B(\nu^\#)}\\ &\quad+3\beta^{-1}\langle AA^*+A^*A\rangle_{H_\Lambda^B(\nu^\#)}.\end{aligned} \tag{3.17}$$

Since we are looking for the estimate of (3.5) (and consequently of (3.7)) from *above*, the inequalities (3.8) and (3.17) show that it remains only to estimate the mean value of the second term in (3.7). Here we formulate the result; the proof is postponed until appendix A.

Theorem 3.7. Let $(\theta,\mu)\in D$ Then there are two non-negative, locally bounded in D functions

$$\begin{aligned}a &= a(\theta,\mu,\nu^\#)\\ b &= b(\theta,\mu,\nu^\#)\end{aligned} \tag{3.18}$$

such that for $|\nu|\leqslant r_0$, $r_0>0$, one has:

$$\langle A^*[A,[H_\Lambda^B(\mu,\nu^\#),A^*]]A\rangle_{H_\Lambda^B(\nu^\#)} \leqslant a\langle A^*A\rangle_{H_\Lambda^B(\nu^\#)} + b. \tag{3.19}$$

To prove the next statement (theorem 3.14) we need first to prove the following lemmas.

Lemma 3.8. For $(\theta,\mu)\in Q$ and $\nu\in\mathbb{C}$ we have

$$p_\Lambda[H_\Lambda^B(\nu^\#)] \leqslant \tilde{p}_\Lambda^I(\beta,\mu) + \left\{\frac{1}{\beta V}\sum_{n_0=0}^{\infty}\mathrm{e}^{\frac{\beta}{2}[(\varphi(0)+2)n_0-v(0)n_0^2/V]}\right\} + |\nu|^2. \tag{3.20}$$

Proof. By the inequality

$$-\sqrt{V}(\overline{\nu}a_0 + \nu a_0^*) \geqslant -a_0^* a_0 - |\nu|^2 V$$

(3.20) follows immediately from the estimate (cf (2.11) and (2.12))

$$H_\Lambda^B(\nu^\#) - \mu N_\Lambda \geqslant \sum_{k\in\Lambda^*, k\neq 0}\left(\varepsilon_k - \mu - \frac{v(k)}{2V}\right) n_k + \frac{v(0)}{2V} n_0^2 - (\mu + \tfrac{1}{2}\varphi(0) + 1) n_0 - |\nu|^2 V.$$

□

Corollary 3.9. By (3.20), in the thermodynamic limit, one gets

$$p^B(\beta, \mu; \nu^\#) \leqslant p^I(\beta, \mu) + \tfrac{1}{2}\sup_{\rho\geqslant 0}[(\varphi(0) + 2)\rho - v(0)\rho^2] + |\nu|^2 \tag{3.21}$$

for $(\theta, \mu) \in Q$, $\nu \in \mathbb{C}$.

Lemma 3.10. For any $\mu < 0$ and $\nu \in \mathbb{C}$ one has the estimate

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leqslant g_\Lambda(\beta, \mu; \nu^\#) < \infty. \tag{3.22}$$

Proof. For any $\mu < 0$ there is $\delta > 0$ such that $\mu + \delta < 0$. Then by the Bogoliubov inequality we obtain

$$\delta \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leqslant p_\Lambda[H_\Lambda^B(\nu^\#) - \delta N_\Lambda] - p_\Lambda[H_\Lambda^B(\nu^\#)]. \tag{3.23}$$

Therefore, by lemma 3.8 one gets (3.22) for

$$g_\Lambda(\beta, \mu; \nu^\#) \equiv \frac{1}{\delta}(p_\Lambda^B(\beta, \mu + \delta; \nu^\#) - p_\Lambda^B(\beta, \mu; \nu^\#)). \tag{3.24}$$

□

Corollary 3.11. In the thermodynamic limit (3.24) gives

$$\rho^B(\beta, \mu; \nu^\#) = \lim_\Lambda \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leqslant \frac{1}{\delta}(p^B(\beta, \mu + \delta; \nu^\#) - p^B(\beta, \mu; \nu^\#)) \equiv g(\beta, \mu; \nu^\#). \tag{3.25}$$

In fact, for $\mu < 0$, $\nu \in \mathbb{C}$, we have that

$$\rho^B(\beta, \mu; \nu^\#) = \partial_\mu p^B(\beta, \mu; \nu^\#) \tag{3.26}$$

by Griffiths' lemma [8].

Corollary 3.12. By virtue of (3.22) one obviously obtains:

$$\left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} \leqslant g_\Lambda(\beta, \mu; \nu^\#); \quad \left| \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \right| = \left| \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \right| \leqslant \sqrt{g_\Lambda(\beta, \mu; \nu^\#)}. \tag{3.27}$$

Remark 3.13. To optimize the estimate (3.4) we have to look for $\sup_{c\in\mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#)$. Since by definition 2.11 and (3.3)

$$H_\Lambda^B(c^\#, \mu, \nu^\#) = H_\Lambda^B(c^\#, \mu) - V(\nu\overline{c} + \overline{\nu}c) \geqslant H_\Lambda^B(c^\#, \mu) - V(|\nu|^2|c|^2 + 1) \tag{3.28}$$

from (2.28) one has that for any $(\theta, \mu) \in Q$ and a fixed $\nu^\#$ there is $A \geqslant 0$ such that

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \leqslant A - \tfrac{1}{2}v(0)|c|^4. \tag{3.29}$$

Thus for any compact $K \subset Q \times \{\nu \in \mathbb{C}\}$, the optimal value of $|c|$ is bounded by a positive constant $M_K < \infty$.

Now we are in position to prove the main statement of this section (see (3.1)) about *exactness* of the Bogoliubov approximation for the WIBG.

Theorem 3.14. Let $(\theta, \mu) \in D$. Then

$$\lim_{\Lambda} \left\{ p_\Lambda^B(\beta, \mu, \nu^\#) - \sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu; c^\#, \nu^\#) \right\} = 0 \tag{3.30}$$

locally uniformly in D for $|\nu| \leqslant r_0$, $r_0 > 0$.

Proof. From the main inequality (3.4) one obtains

$$0 \leqslant \inf_{c\in\mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) \equiv \Delta_\Lambda(\beta, \mu; \hat{c}_\Lambda^\#(\beta, \mu, \nu^\#), \nu^\#) \leqslant \frac{1}{V} \langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) \rangle_{H_\Lambda^B(\nu^\#)}. \tag{3.31}$$

By virtue of (3.5)–(3.7), estimates (3.6), (3.8), (3.11), (3.17), (3.19) and remark 3.13, there are positive constants u and w independent of the volume V, such that

$$\frac{1}{V} \langle H_\Lambda^B(c^\#, \mu, \nu^\#) - H_\Lambda^B(\mu, \nu^\#) \rangle_{H_\Lambda^B(\nu^\#)} \leqslant u + \frac{w}{2} \langle \{(a_0^* - \sqrt{V} c^*), (a_0 - \sqrt{V} c)\} \rangle_{H_\Lambda^B(\nu^\#)} \tag{3.32}$$

locally uniformly in D.

Put $c \equiv \langle a_0/\sqrt{V} \rangle_{H_\Lambda^B(\nu^\#)}$ which is bounded (see (3.27)). Then

$$\Delta_\Lambda(\beta, \mu; \hat{c}_\Lambda^\#, \nu^\#) \leqslant \Delta_\Lambda(\beta, \mu; \langle a_0^\#/\sqrt{V} \rangle_{H_\Lambda^B(\nu^\#)}, \nu^\#)$$

and estimates (3.31) and (3.32) give

$$0 \leqslant \inf_{c\in\mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) \leqslant \frac{u}{V} + \frac{w}{2V} \langle \{(a_0^* - \langle a_0^* \rangle), (a_0 - \langle a_0 \rangle)\} \rangle_{H_\Lambda^B(\nu^\#)} \tag{3.33}$$

where for short $\langle a_0^\# \rangle \equiv \langle a_0^\# \rangle_{H_\Lambda^B(\nu^\#)}$. Let $\delta a_0^\# \equiv a_0^\# - \langle a_0^\# \rangle$. Then, by the Harris inequality (see [9, 14]) one obtains

$$\tfrac{1}{2} \langle \{\delta a_0^*, \delta a_0\} \rangle_{H_\Lambda^B(\nu^\#)} \leqslant (\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} + \frac{\beta}{12} \langle [\delta a_0^*, [H_\Lambda^B(\mu, \nu^\#), \delta a_0]] \rangle_{H_\Lambda^B(\nu^\#)}. \tag{3.34}$$

By condition (B) on the interaction and lemma 3.10 we have:

$$\begin{aligned} \langle [\delta a_0^*, [H_\Lambda^B(\mu, \nu^\#), \delta a_0]] \rangle_{H_\Lambda^B(\nu^\#)} &= \left\langle \frac{v(0)}{V} N_\Lambda - \mu + \frac{1}{V} \sum_{k\in\Lambda^*} v(k) a_k^* a_k \right\rangle_{H_\Lambda^B(\nu^\#)} \\ &\leqslant 2v(0) g_\Lambda(\beta, \mu; \nu^\#) - \mu. \end{aligned} \tag{3.35}$$

Since by (2.6), (2.10) and (3.25) we have a uniform boundedness: $g_\Lambda(\beta, \mu; \nu^\#) < g_0$ for each compact $C_0(\mu < 0) \subset D$ and $|\nu| \leqslant r_0$, the estimate (3.33) in this compact set takes the form:

$$0 \leqslant \inf_{c\in\mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) \leqslant \frac{1}{V} [\tilde{u} + w(\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)}]. \tag{3.36}$$

Now we can proceed with the standard reasoning of the approximation Hamiltonian method (see [9]). First we note that

$$(\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} = \frac{1}{\beta} \partial_\nu \partial_{\bar{\nu}} p_\Lambda [H_\Lambda^B(\nu^\#)]. \tag{3.37}$$

By the (canonical) gauge transformation $a_0 \to a_0 \mathrm{e}^{\mathrm{i}\varphi}$, $\varphi = \arg \nu$, one finds that in fact

$$p_\Lambda[H_\Lambda^B(\nu^\#)] = p_\Lambda^B(\beta, \mu; |\nu| \equiv r).$$

Then passing in (3.37) to polar coordinates (r, φ) we obtain:

$$(\delta a_0^*, \delta a_0)_{H_\Lambda^B(\nu^\#)} = \frac{1}{4\beta r}\partial_r(r\partial_r p_\Lambda^B). \tag{3.38}$$

Let $c = |c|\mathrm{e}^{\mathrm{i}\psi}$, $\psi = \arg c$. Then by (3.3), (3.4) one obtains

$$\inf_{c\in\mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) = \inf_{|c|,\psi} \Delta_\Lambda(\beta, \mu; |c|\mathrm{e}^{\pm\mathrm{i}\psi}, r\mathrm{e}^{\pm\mathrm{i}\varphi}) = \inf_{|c|} \hat{\Delta}_\Lambda(\beta, \mu; |c|\mathrm{e}^{\pm\mathrm{i}\varphi}, r)$$
$$\equiv \inf_{|c|} \tilde{\Delta}_\Lambda(r). \tag{3.39}$$

Therefore, by (3.36)

$$\int_R^{R+\varepsilon} r \inf_{|c|} \tilde{\Delta}_\Lambda(r)\,\mathrm{d}r \leqslant \frac{1}{V}\left\{\tilde{u}\frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{4\beta}(r\partial_r p_\Lambda^B)|_R^{R+\varepsilon}\right\} \tag{3.40}$$

for $[R, R+\varepsilon] \subset [0, r_0]$. Note that by (3.27) we have

$$\partial_r p_\Lambda^B = 2|\langle a_0/\sqrt{V}\rangle_{H_\Lambda^B(\nu^\#)}| \leqslant 2g_0^{\frac{1}{2}} \qquad (\theta, \mu) \in C_0 \subset D, |\nu| \leqslant r_0. \tag{3.41}$$

Therefore, (3.40) takes the form

$$\int_R^{R+\varepsilon} r \inf_{|c|} \tilde{\Delta}_\Lambda(r)\,\mathrm{d}r \leqslant \frac{1}{V}\left\{\tilde{u}\frac{(R+\varepsilon)^2 - R^2}{2} + \frac{w}{2\beta}g_0^{\frac{1}{2}}(2R+\varepsilon)\right\}. \tag{3.42}$$

Since by corollary 3.12 and remark 3.13

$$\left|\partial_r \inf_{|c|} \tilde{\Delta}_\Lambda(r)\right| \leqslant 2g_\Lambda^{\frac{1}{2}} + 2|\hat{c}_\Lambda| \leqslant 2(g_0^{\frac{1}{2}} + M)$$

for $r \in [R, R+\varepsilon]$ we obtain:

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R) \leqslant \inf_{|c|} \tilde{\Delta}_\Lambda(r) + 2(r - R)(g_0^{\frac{1}{2}} + M).$$

Hence,

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R)\frac{(R+\varepsilon)^2 - R^2}{2} \leqslant \int_R^{R+\varepsilon} r \inf_{|c|} \tilde{\Delta}_\Lambda(r)\,\mathrm{d}r + 2(g_0^{\frac{1}{2}} + M)\left(\frac{r^3}{3} - R\frac{r^2}{2}\right)\bigg|_R^{R+\varepsilon}.$$

Then by (3.42) we obtain

$$\inf_{|c|} \tilde{\Delta}_\Lambda(R) \leqslant \frac{1}{V}\left\{\tilde{u} + \frac{w}{\beta}g_0^{\frac{1}{2}}\varepsilon^{-1}\right\} + (g_0^{\frac{1}{2}} + M)\varepsilon\frac{R + \frac{2}{3}\varepsilon}{R + \frac{1}{2}\varepsilon}. \tag{3.43}$$

Note that $\varepsilon > 0$ is still arbitrary. Minimizing the right-hand side of (3.43) one obtains that for large V the optimal value of $\varepsilon \sim 1/\sqrt{V}$. Hence, for $V \to \infty$ one gets from (3.43) the asymptotic estimate

$$0 \leqslant \inf_{c\in\mathbb{C}} \Delta_\Lambda(\beta, \mu; c^\#, \nu^\#) \leqslant \delta_\Lambda \equiv \text{constant } \frac{1}{\sqrt{V}} \tag{3.44}$$

valid for each compact $C_0 \subset D$ and $|\nu| \leqslant r_0$. One gets (3.30) for $(\theta, \mu) \in D$ by extension of (3.43) to $\mu = 0$ by continuity. □

Corollary 3.15. Let $(\theta, \mu) \in D$. Then, if one considers the Bogoliubov approximation for the statistical operator W_Λ

$$W_\Lambda = \mathrm{e}^{-\beta H_\Lambda^B(\mu,\nu^\#)}$$

we have

$$\lim_{\Lambda} \left\{ p_{\Lambda}^{B}(\beta, \mu, \nu^{\#}) - \sup_{c \in \mathbb{C}} p_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \right\} = 0 \tag{3.45}$$

locally uniformly in D, where

$$p_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \equiv \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}'_{\Lambda}} W_{\Lambda}(c^{\#}) \tag{3.46}$$

$|\nu| \leqslant r_0$, $r_0 > 0$, and $W_{\Lambda}(c^{\#})$ is defined by (2.24).

Proof. Using for calculation of $\mathrm{Tr}_{\mathcal{F}_{\Lambda}}(-) = \mathrm{Tr}_{\mathcal{F}_{0\Lambda} \otimes \mathcal{F}'_{\Lambda}}(-)$ a product-basis in $\mathcal{F}_{\Lambda}$, one gets (cf definition 2.11)

$$\mathrm{Tr}_{\mathcal{F}_{\Lambda}}(W_{\Lambda}) \geqslant \sup_{\{\psi'_n\}_n} \sum_{n} (\psi_{0\Lambda}(c) \otimes \psi'_n, \mathrm{e}^{-\beta H_{\Lambda}^{B}(\mu, \nu^{\#})} \psi_{0\Lambda}(c) \otimes \psi'_n) \equiv \mathrm{Tr}_{\mathcal{F}'_{\Lambda}} W_{\Lambda}(c^{\#})$$

where $\{\psi'_n\}_n$ is an arbitrary orthonormal basis in $\mathcal{F}'_{\Lambda}$. Now, whenever $\psi_{0\Lambda}(c) \otimes \psi'_n$ are in the form-domain of $H_{\Lambda}^{B}(\mu, \nu^{\#})$, the Peierls inequality [3] gives (by definition of $H_{\Lambda}^{B}(\mu, c^{\#}, \nu^{\#})$, see (2.24)) that

$$(\psi_{0\Lambda}(c) \otimes \psi'_n, \mathrm{e}^{-\beta H_{\Lambda}^{B}(\mu, \nu^{\#})} \psi_{0\Lambda}(c) \otimes \psi'_n) \geqslant \mathrm{e}^{-\beta(\psi'_n, H_{\Lambda}^{B}(c^{\#}, \mu, \nu^{\#})\psi'_n)}.$$

Therefore, one obtains

$$\tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \leqslant p_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \leqslant p_{\Lambda}^{B}(\beta, \mu, \nu^{\#}). \tag{3.47}$$

From (3.47) we deduce by theorem 3.14 the thermodynamic limit (3.45). □

Corollary 3.16. Since the variational pressure $\tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#})$ is known in the explicit form (see (2.27), (2.28) and (3.3)):

$$\tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) = \tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}) + (\nu\bar{c} + \bar{\nu}c) \tag{3.48}$$

the following thermodynamic limits exist:

$$\begin{aligned} &\tilde{p}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) = \lim_{\Lambda} \tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \\ &\tilde{p}^{B}(\beta, \mu; \hat{c}^{\#}(\beta, \mu; \nu^{\#}), \nu^{\#}) = \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}, \nu^{\#}) \right] = \sup_{c \in \mathbb{C}} \tilde{p}^{B}(\beta, \mu; c^{\#}, \nu^{\#}). \end{aligned} \tag{3.49}$$

Then by virtue of the locally uniform estimate (3.44) and of extension by continuity to $\mu = 0$ we get

$$p^{B}(\beta, \mu; \nu^{\#}) = \lim_{\Lambda} p_{\Lambda}[H_{\Lambda}^{B}(\nu^{\#})] = \sup_{c \in \mathbb{C}} \tilde{p}^{B}(\beta, \mu; c^{\#}, \nu^{\#}). \tag{3.50}$$

For $(\theta, \mu) \in D$, $|\nu| \leqslant r_0$ and (cf (3.1)) the limit $|\nu| \to 0$:

$$p^{B}(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^{B}(\beta, \mu; c^{\#}). \tag{3.51}$$

Corollary 3.17. Inequalities (2.25) and (2.30) give

$$p^{I}(\beta, \mu) \leqslant \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \tilde{p}_{\Lambda}^{B}(\beta, \mu; c^{\#}) \right] \leqslant p^{B}(\beta, \mu).$$

Then definitions (2.36), (2.42) imply $D_0 \subseteq D$, whereas (3.30) implies that $D_0 = D$, which proves (3.2). Hence, we have

$$p^{B}(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^{B}(\beta, \mu; c^{\#}) \qquad \text{for } (\theta, \mu) \in Q \backslash \partial D. \tag{3.52}$$

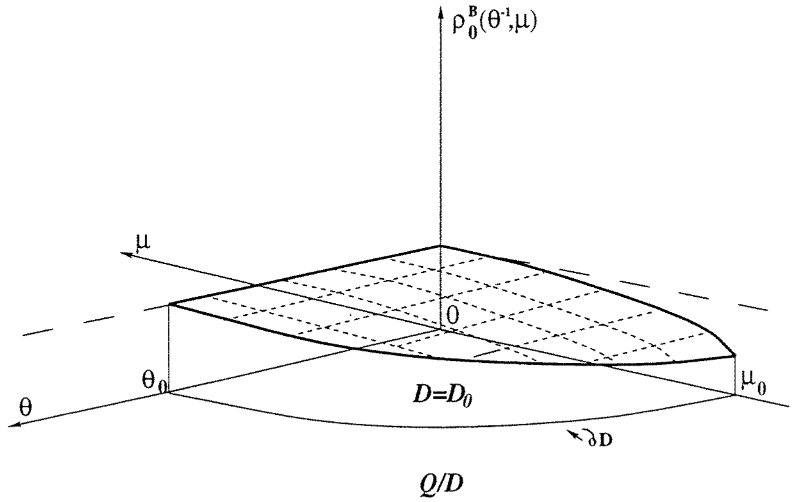

Figure 3. Discontinuous behaviour of the Bose condensate density $|\hat{c}(\beta,\mu)|^2 = \rho_0^B(\theta^{-1},\mu)$ for the Bogoliubov WIBG: $\rho_0^B > 0$ in domain $D = D_0$ and $\rho_0^B = 0$ in the rest of the stability domain $Q\backslash\overline{D}$.

Remark 3.18. Since (2.28) implies that

$$\tilde{p}_\Lambda^B(\beta,\mu;c^\# = 0) = p_\Lambda^I(\beta,\mu) \tag{3.53}$$

by (2.36), (2.46) and (3.2) we get

$$D_0 = \{(\theta,\mu) : |\hat{c}(\beta,\mu;\nu)| > 0\} = \{(\theta,\mu) : \rho_0^B(\beta,\mu) > 0\} = D. \tag{3.54}$$

Therefore, (see remark 2.18) the condition (C) is sufficient and necessary for $D \neq \{\emptyset\}$.

4. Thermodynamics of the weakly imperfect Bose gas

Since the pressure $\tilde{p}_\Lambda^B$ (2.28) and $\lim_\Lambda \tilde{p}_\Lambda^B = \tilde{p}^B$ are known explicitly:

$$\tilde{p}^B(\beta,\mu;c^\#,\nu^\#) = \frac{1}{\beta(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k \, \ln(1-\mathrm{e}^{-\beta E_k(|c|^2)})^{-1} - \frac{1}{\beta(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k \, [E_k(|c|^2) - f_k(|c|^2)] + \mu|c|^2 - \tfrac{1}{2}v(0)|c|^4 + (\nu\bar{c} + \bar{\nu}c) \tag{4.1}$$

theorem 3.14 and corollaries 3.16 and 3.17 give an exact solution of the model (1.4) on the level of thermodynamics. Therefore, (3.52) gives access to the thermodynamic properties of the WIBG for all $(\theta,\mu) \in Q$ except the line of transitions ∂D (see figures 1 and 3).

The aim of this section is to discuss thermodynamic properties of the model (1.4) and in particular the Bose condensate which appears in domain D. The first statement concerns the gauge symmetry-breaking in domain D.

Theorem 4.1. Let $D \neq \{\emptyset\}$. Then quasi-averages

$$\lim_{\{\nu\to 0:\arg\nu=\varphi\}} \lim_\Lambda \langle a_0^\#/\sqrt{V}\rangle_{H_\Lambda^B(\nu^\#)} = \mathrm{e}^{\pm\mathrm{i}\varphi}|\hat{c}(\beta,\mu)| = \left\{ \begin{array}{l} \neq 0,\, (\theta,\mu) \in D \\ = 0,\, (\theta,\mu) \in Q\backslash\overline{D} \end{array} \right\}. \tag{4.2}$$

Proof. As in the proof of theorem 3.14 by the gauge transformation

$$\mathcal{U}_\varphi a_0 \mathcal{U}_\varphi^* = a_0 \mathrm{e}^{-\mathrm{i}\varphi} = \tilde{a}_0 \qquad \varphi = \arg\nu$$

we get

$$\begin{aligned}\tilde{H}^B_\Lambda(\mu, r) &= \mathcal{U}_\varphi H^B_\Lambda(\mu, \nu^\#)\mathcal{U}^*_\varphi = \tilde{H}^B_\Lambda - \mu\tilde{N}_\Lambda - \sqrt{V}r(\tilde{a}_0 + \tilde{a}^*_0) \\ p_\Lambda[H^B_\Lambda(\nu^\#)] &= p_\Lambda[\mathcal{U}_\varphi H^B_\Lambda(\mu, \nu^\#)\mathcal{U}^*_\varphi] = p^B_\Lambda(\beta, \mu; r = |\nu|).\end{aligned} \tag{4.3}$$

By virtue of

$$0 = \langle[\tilde{H}^B_\Lambda(\mu, r), \tilde{N}_\Lambda]\rangle_{\tilde{H}^B_\Lambda(r)} = r\sqrt{V}\langle\tilde{a}_0 - \tilde{a}^*_0\rangle_{\tilde{H}^B_\Lambda(r)}$$

and (cf (3.10))

$$0 \leqslant \langle[\tilde{N}_\Lambda, [\tilde{H}^B_\Lambda(\mu, r), \tilde{N}_\Lambda]]\rangle_{\tilde{H}^B_\Lambda(r)} = r\sqrt{V}\langle\tilde{a}_0 + \tilde{a}^*_0\rangle_{\tilde{H}^B_\Lambda(r)}$$

we obtain

$$\langle\tilde{a}_0\rangle_{\tilde{H}^B_\Lambda(r)} = \langle\tilde{a}^*_0\rangle_{\tilde{H}^B_\Lambda(r)} \geqslant 0. \tag{4.4}$$

Since (cf (3.9))

$$\begin{aligned}&\partial^2_r p^B_\Lambda(\beta, \mu; r) = \beta(\{(\tilde{a}_0 + \tilde{a}^*_0) - \langle\tilde{a}_0 + \tilde{a}^*_0\rangle_{\tilde{H}^B_\Lambda(r)}\} \\ &\{(\tilde{a}_0 + \tilde{a}^*_0) - \langle\tilde{a}_0 + \tilde{a}^*_0\rangle_{\tilde{H}^B_\Lambda(r)}\})_{\tilde{H}^B_\Lambda(r)} \geqslant 0\end{aligned} \tag{4.5}$$

by theorem 3.14 and corollary 3.16 the sequence of the convex (for $r \geqslant 0$) functions $\{p^B_\Lambda(\beta, \mu; r)\}_\Lambda$ converges to the (convex function)

$$\begin{aligned}\hat{p}^B(\beta, \mu; r) &\equiv \sup_{c\in\mathbb{C}} \tilde{p}^B(\beta, \mu; c^\#, \nu^\#) = \sup_{\substack{|c|\geqslant 0\\ \psi=\arg c}} \tilde{p}^B(\beta, \mu; |c|\mathrm{e}^{\pm\mathrm{i}\psi}, |\nu|\mathrm{e}^{\pm\mathrm{i}\varphi}) \\ &= \tilde{p}^B(\beta, \mu; |\hat{c}(\beta, \mu; r)|\mathrm{e}^{\pm\mathrm{i}\varphi}, |\nu|\mathrm{e}^{\pm\mathrm{i}\varphi})\end{aligned} \tag{4.6}$$

see (3.38) and (4.1), locally uniformly in $D \times [0, r_0]$. By explicit calculations one finds that the derivatives

$$\begin{aligned}0 &\leqslant \partial_r \hat{p}^B(\beta, \mu; r) = 2|\hat{c}(\beta, \mu; r)| \leqslant C_1 \\ 0 &\leqslant \partial^2_r \hat{p}^B(\beta, \mu; r) = 2\partial_r|\hat{c}(\beta, \mu; r)| \leqslant C_2\end{aligned} \tag{4.7}$$

are continuous and bounded in $D \times [0, r_0]$. Therefore, by Griffiths' lemma [8]

$$\lim_\Lambda \partial_r p_\Lambda[\tilde{H}^B_\Lambda(r)] = \lim_\Lambda \left\langle\frac{\tilde{a}_0 + \tilde{a}^*_0}{\sqrt{V}}\right\rangle_{\tilde{H}^B_\Lambda(r)} = 2|\hat{c}(\beta, \mu; r)|$$

or by (4.4),

$$\begin{aligned}&\lim_\Lambda\langle\tilde{a}_0/\sqrt{V}\rangle_{\tilde{H}^B_\Lambda(r)} = |\hat{c}(\beta, \mu; r)| \\ &\lim_\Lambda\langle\tilde{a}^*_0/\sqrt{V}\rangle_{\tilde{H}^B_\Lambda(r)} = |\hat{c}(\beta, \mu; r)|.\end{aligned} \tag{4.8}$$

Returning in (4.8) back to original creation/annihilation operators, one obtains

$$\begin{aligned}&\lim_\Lambda\langle a_0/\sqrt{V}\rangle_{H^B_\Lambda(\nu^\#)} = \mathrm{e}^{+\mathrm{i}\varphi}|\hat{c}(\beta, \mu; r)| \\ &\lim_\Lambda\langle a^*_0/\sqrt{V}\rangle_{H^B_\Lambda(\nu^\#)} = \mathrm{e}^{-\mathrm{i}\varphi}|\hat{c}(\beta, \mu; r)|.\end{aligned} \tag{4.9}$$

Then the first part of the statement (4.2) follows from (4.9) and the continuity of the solution $\hat{c}(\beta, \mu; r)$ at $r = 0$, while the second part follows from (3.54). □

Corollary 4.2. Notice that by the gauge invariance

$$\left\langle \frac{a_0^\#}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#=0)} = 0. \tag{4.10}$$

Therefore, we have the gauge symmetry-breaking:

$$\lim_{\nu\to 0}\lim_{\Lambda}\left\langle \frac{a_0^\#}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \neq \lim_{\Lambda}\lim_{\nu\to 0}\left\langle \frac{a_0^\#}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \tag{4.11}$$

as soon as the Bose condensation $\rho_0^B(\beta,\mu) \neq 0$.

Corollary 4.3. Since by (4.5), (4.7)

$$\partial_r^2\Big(\inf_{|c|}\tilde{\Delta}_\Lambda(r)\Big) = \partial_r^2(p_\Lambda^B(\beta,\mu;r) - \hat{p}^B(\beta,\mu;r)) \geqslant -C_2$$

the Kolmogorov lemma [15] implies that

$$\left|\left\langle \frac{\tilde{a}_0}{\sqrt{V}} \right\rangle_{\tilde{H}_\Lambda^B(r)} - |\hat{c}_\Lambda(\beta,\mu;r)|\right| \leqslant 2\sqrt{\delta_\Lambda C_2} \tag{4.12}$$

for $r \in [l_\Lambda, r_0 - l_\Lambda]$, $l_\Lambda = 2\sqrt{\delta_\Lambda/C_2}$ (see (3.44) and (4.8)).

Note that the Cauchy–Shwartz inequality gives

$$\left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_\Lambda^B(\nu^\#)} \leqslant \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B(\nu^\#)}.$$

Hence, by (2.45) and (4.2) one gets

$$|\hat{c}_\Lambda(\beta,\mu)|^2 \leqslant \lim_{\nu\to 0}\lim_{\Lambda}\left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B(\nu^\#)} = \rho_0^B(\beta,\mu) \tag{4.13}$$

which is in coherence with the definitions of domains D_0 and D (cf theorem 2.16 and corollary 2.17). To prove equality in (4.13) we proceed as follows.

Theorem 4.4. Let

$$\begin{aligned} H_{\Lambda,\alpha}^B &= H_\Lambda^B + \alpha a_0^* a_0 \\ H_{\Lambda,\alpha}^B(\nu^\#) &= H_{\Lambda,\alpha}^B - \sqrt{V}(\nu a_0^* + \overline{\nu} a_0) \end{aligned} \tag{4.14}$$

for $\alpha \in \mathbb{R}^1$. Then

$$p_\alpha^B(\beta,\mu;\nu^\#) = \lim_\Lambda p_\Lambda[H_{\Lambda,\alpha}^B(\nu^\#)] = \lim_\Lambda\left[\sup_{c\in\mathbb{C}} \tilde{p}_{\Lambda,\alpha}^B(\beta,\mu;c^\#,\nu^\#)\right] \tag{4.15}$$

for $|\nu| \leqslant r_0, r_0 > 0$ and $(\theta,\mu) \in Q\backslash\partial D_\alpha$ where domain

$$D_\alpha \equiv \{(\theta,\mu) : p_\alpha^B(\beta,\mu;\nu^\# = 0) > p^I(\beta,\mu)\}. \tag{4.16}$$

Remark 4.5. Since $H^B_{\Lambda,\alpha=\frac{1}{2}\varphi(0)} = \hat{H}_\Lambda^B$ (see (2.47)), by theorem 2.20 we find that $D_{\alpha=\frac{1}{2}\varphi(0)} = \{\emptyset\}$.

Our reasoning below is a translation of some results of sections 2 and 3 to the perturbed Hamiltonian $H_{\Lambda,\alpha}^B$ for small α.

Lemma 4.6. If potential $v(k)$ satisfies (A)–(C), then

$$D_{0\alpha} \equiv \left\{(\theta,\mu) : \sup_{c\in\mathbb{C}} \tilde{p}_\alpha^B(\beta,\mu;c^\#) > p^I(\beta,\mu)\right\} \neq \{\emptyset\}. \tag{4.17}$$

for $\alpha < -\mu_0$, where μ_0 is defined by lemma 2.15.

Proof. Since the $\eta_{\Lambda,\alpha}(\mu; x)$ for the Hamiltonian (4.14) (cf (2.28)) has the form

$$\eta_{\Lambda,}(\mu; x) = -\frac{1}{2V}\sum_{k\in\Lambda^*,k\neq 0}(E_k - f_k) + (\mu - \alpha)x - \frac{1}{2}v(0)x^2 \tag{4.18}$$

one can follow the line of reasoning in the proofs of lemma 2.15 and theorem 2.16 to find (4.17) for $\mu \leqslant 0$ such that $(\mu-\alpha) > \mu_0$. Therefore, the value of $\mu_0+\alpha$ must be negative. □

By continuity from (4.18) with respect to α it is clear that $\lim_{\alpha\to 0} D_{0\alpha} = D_0$. Now we turn to the proof of theorem 4.4.

Proof of theorem 4.4. (a) Since the Bogoliubov approximation (2.24) gives the estimate of the pressure $p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\#)]$ from below (see proposition 2.12) as:

$$\sup_{c\in\mathbb{C}} \tilde{p}^B_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) \leqslant p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\#)]$$

by the Bogoliubov inequality we get (cf (3.4)):

$$\begin{aligned} 0 &\leqslant \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) \equiv p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\#)] - \tilde{p}^B_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) \\ &\leqslant \frac{1}{V}\langle H^B_{\Lambda,\alpha}(\hat{c}^\#, \mu, \nu^\#) - H^B_{\Lambda,\alpha}(\mu, \nu^\#)\rangle_{H^B_{\Lambda,\alpha}(\nu^\#)}. \end{aligned} \tag{4.19}$$

(b) For operators $A^\# \equiv a_0^\# - \sqrt{V}c^\#$ and for a Taylor expansion of $H^B_{\Lambda,\alpha}(\hat{c}^\#, \mu, \nu^\#)$ around $a_0^\#$ one obtains the estimate

$$\begin{aligned} 0 &\leqslant \inf_{c\in\mathbb{C}} \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) = \Delta_{\Lambda,\alpha}(\beta, \mu; \hat{c}^\#_{\Lambda,\alpha}(\beta, \mu; \nu^\#), \nu^\#) \\ &\leqslant u_\alpha + \frac{w_\alpha}{2}\langle\{(a_0^* - \sqrt{V}\bar{c}), (a_0 - \sqrt{V}c)\}\rangle_{H^B_\Lambda(\nu^\#)} \end{aligned} \tag{4.20}$$

by repeating verbatim the arguments developed from remark 3.2 through to remark 3.13. The only difference with the case $\alpha = 0$ comes from

$$[A, [H^B_{\Lambda,\alpha}(\mu, \nu^\#), A^*]] = [A, [H^B_\Lambda(\mu, \nu^\#), A^*]] + \alpha$$

cf (3.35), and the note that $\lim_{\alpha\to 0} u_\alpha = u$ and $\lim_{\alpha\to 0} w_\alpha = w$.

(c) Put $c^\# \equiv \langle a_0^\#/\sqrt{V}\rangle_{H^B_{\Lambda,\alpha}(\nu^\#)}$ in the left-hand side of (4.20). The same line of reasoning as in theorem 3.14 gives the asymptotic estimate

$$0 \leqslant \inf_{c\in\mathbb{C}} \Delta_{\Lambda,\alpha}(\beta, \mu; c^\#, \nu^\#) \leqslant \delta_{\Lambda,\alpha} \equiv \text{ constant } (\alpha)\frac{1}{\sqrt{V}} \tag{4.21}$$

valid for $(\theta, \mu) \in Q\backslash\partial D_\alpha$, $|\alpha| < -\mu_0$, and $|\nu| \leqslant r_0$ which ensures the proof of (4.15) for $D_\alpha \neq \{\emptyset\}$.

Corollary 4.7. Since

$$\partial^2_\alpha p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\#)] = \frac{\beta}{V}((a_0^*a_0 - \langle a_0^*a_0\rangle_{H^B_{\Lambda,\alpha}(\nu^\#)}), (a_0^*a_0 - \langle a_0^*a_0\rangle_{H^B_{\Lambda,\alpha}(\nu^\#)}))_{H^B_{\Lambda,\alpha}(\nu^\#)} \geqslant 0$$

functions $\{p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\# = 0)]\}_\Lambda$ are convex for $\alpha \in \mathbb{R}^1$. The same is obviously true (cf (4.1), (4.14) and (4.15)) for the limit

$$\begin{aligned} \lim_\Lambda p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\# = 0)] &= \sup_{c\in\mathbb{C}} \tilde{p}^B_\alpha(\beta, \mu; c^\#, \nu^\# = 0) = \tilde{p}^B_\alpha(\beta, \mu; \hat{c}^\#_\alpha(\beta, \mu), 0) \\ &= \tilde{p}^B(\beta, \mu; \hat{c}^\#_\alpha(\beta, \mu), 0) - \alpha|\hat{c}_\alpha(\beta, \mu)|^2. \end{aligned} \tag{4.22}$$

By explicit calculations one finds that

$$\partial_\alpha \tilde{p}^B_\alpha(\beta,\mu;\hat{c}^\#_\alpha(\beta,\mu),0) = -|\hat{c}_\alpha(\beta,\mu)|^2 < \text{ constant} \tag{4.23}$$

for $(\theta,\mu)\in Q$ and $|\alpha|\leqslant -\mu_0$. Therefore, by Griffiths' lemma [8] we obtain:

$$\lim_\Lambda \partial_\alpha p_\Lambda[H^B_{\Lambda,\alpha}(\nu^\#=0)] = \lim_\Lambda \left(-\left\langle \frac{a_0^* a_0}{V}\right\rangle_{H^B_{\Lambda,\alpha}(\nu^\#=0)}\right) = -|\hat{c}_\alpha(\beta,\mu)|^2. \tag{4.24}$$

Corollary 4.8. By the continuity in $\alpha\to 0$, equations (4.2) and (4.24) imply that

$$\rho^B_0(\beta,\mu) = \lim_\Lambda \left\langle \frac{a_0^* a_0}{V}\right\rangle_{H^B_\Lambda} = \lim_\Lambda \left\langle \frac{a_0^*}{\sqrt{V}}\right\rangle_{H^B_\Lambda} \lim_\Lambda \left\langle \frac{a_0}{\sqrt{V}}\right\rangle_{H^B_\Lambda} = |\hat{c}(\beta,\mu)|^2. \tag{4.25}$$

We conclude this section by analysis of the Bose condensate $\rho^B_0(\beta,\mu)$ behaviour. By virtue of (4.25) it reduces to the analysis of the behaviour of $|\hat{c}(\beta,\mu)|$ which corresponds to the $\sup_{c\in\mathbb{C}}$ of the trial pressure (4.1):

$$\tilde{p}^B(\beta,\mu;c^\#,\nu^\#=0) = \xi(\beta,\mu;x\equiv|c|^2) + \eta(\mu;x\equiv|c|^2) \equiv \tilde{p}^B(\beta,\mu;c^\#) \tag{4.26}$$

where (cf. (2.28) and (2.29))

$$\begin{aligned}
\xi(\beta,\mu;x) &= \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3} \mathrm{d}^3k\, \ln(1-\mathrm{e}^{-\beta E_k})^{-1} \\
\eta(\mu;x) &= \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k\,(f_k - E_k) + \mu x - \tfrac{1}{2}v(0)x^2 \\
f_k &= \varepsilon_k - \mu + x[v(0)+v(k)] \qquad h_k = x v(k) \qquad E_k = \sqrt{f_k^2 - h_k^2}.
\end{aligned} \tag{4.27}$$

Below we collect some properties of the trial pressure (4.26).

(1) For $\mu\leqslant 0$ the function (4.26) is differentiable with respect to $x=|c|^2\geqslant 0$ and

$$\lim_{|c|^2\to\infty} \tilde{p}^B(\beta,\mu;c^\#) = -\infty. \tag{4.28}$$

Hence, $\sup_{x\geqslant 0}(\xi+\eta)(\beta,\mu;x)$ is attained either at $x=0$, or at a positive solution of the equation

$$\begin{aligned}
0 = \partial_x(\xi+\eta)(\beta,\mu;x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k\,(1-\mathrm{e}^{\beta E_k})^{-1}\partial_x E_k \\
&\quad -\frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3k\,(\partial_x E_k - \partial_x f_k) + \mu - x v(0)
\end{aligned} \tag{4.29}$$

—see figure 2.

(2) By definitions (4.27) and the properties (A) and (B) of the potential $v(k)$ one obtains

$$\partial_x f_k = v(0)+v(k) \qquad \partial_x E_k = E_k^{-1}(f_k v(0) + (f_k - h_k)v(k)) \geqslant 0$$

for $\mu\leqslant 0$, $x\geqslant 0$ and any $k\in\mathbb{R}^3$. Therefore, by (4.29) we have

$$\partial_x \tilde{p}^B(\beta,\mu;c^\#=0) \leqslant \partial_x\eta(\mu;x=0) \equiv \partial_x\tilde{p}^B(\beta=\infty,\mu;c^\#=0) = \mu. \tag{4.30}$$

(3) By explicit calculation one finds that $\partial_\mu\partial_x\eta(\mu;x)\geqslant 0$ for $\mu\leqslant 0$ and $x\geqslant 0$. Hence

$$\partial_x\eta(\mu;x) \leqslant \partial_x\eta(\mu=0;x) \tag{4.31}$$

and $\partial_x\eta(\mu=0;x)$ is a concave function of $(0,\infty)$.

(4) Now, let potential $v(k)$ satisfy condition (C). Then

$$\partial_x^2\eta(\mu=0;x) = -v(0) + \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \frac{[v(k)]^2}{\varepsilon_k}\,\mathrm{d}^3k \geqslant 0 \tag{4.32}$$

Since $\eta(\mu = 0; x = 0) = 0$, (4.32) means that the trial pressure

$$\tilde{p}^B(\beta = \infty, \mu; c^\#) = \eta(\mu = 0; x)$$

attains $\sup_{x \geqslant 0}$ for $\hat{x}(\theta = 0, \mu = 0) > 0$, and, by continuity for $(\theta \geqslant 0, \mu \leqslant 0)$, the domain

$$D_0 = \{(\theta, \mu) : \hat{x}(\theta, \mu) > 0\} \neq \{\emptyset\}$$

—see lemma 2.15, theorem 2.16 and figure 2.

(5) Fix $\mu \in D_0$ and $\theta = 0$. Then, according to (4.30),

$$\partial_x \tilde{p}^B(\beta = \infty, \mu; c^\# = 0) = \mu \leqslant 0.$$

But $\partial_x^2 \tilde{p}^B(\beta = \infty, \mu; c^\#, \nu^\# = 0) > 0$ ensures $|\hat{c}(\beta = \infty, \mu)|^2 = \hat{x}(\theta = 0, \mu) \equiv \overline{x}(\mu) > 0$ (see figure 2), i.e.

$$\tilde{p}^B(\beta = \infty, \mu; c^\# = 0) < \tilde{p}^B(\beta = \infty, \mu; |\hat{c}(\beta = \infty, \mu)|^2). \tag{4.33}$$

(6) Since $\partial_x \xi(\beta, \mu; x) < 0$ (see (4.29)) and

$$\partial_\theta \partial_x \xi(\beta, \mu; x) = \frac{(-1)}{(2\pi)^3} \int_{\mathbb{R}^3} \mathrm{d}^3 k \, \frac{\beta^2 E_k \mathrm{e}^{\beta E_k}}{(1 - \mathrm{e}^{\beta E_k})^2} \partial_x E_k < 0 \tag{4.34}$$

there is a critical temperature $\theta_0(\mu)$ (cf theorem 2.16) such that for $\mu \in D_0$ and $\theta = \theta_0(\mu)$, one obtains:

$$\begin{aligned} \sup_{x \geqslant 0} [\xi(\beta_0(\mu), \mu; x) + \eta(\mu; x)] &= \xi(\beta_0(\mu), \mu; 0) + \eta(\mu; 0) \\ &= \xi(\beta_0(\mu), \mu; \hat{x}(\theta_0(\mu), \mu) > 0) + \eta(\mu; \hat{x}(\theta_0(\mu), \mu) > 0) \end{aligned} \tag{4.35}$$

whereas for $\theta < \theta_0(\mu)$ the supremum is attained at $x = \hat{x}(\theta, \mu) > 0$ and for $\theta > \theta_0(\mu)$ it 'jumps' to $\hat{x}(\theta, \mu) = 0$ (see figures 2 and 3). Therefore, we have proved the following statement.

Theorem 4.9. If interaction potential $v(k)$ satisfies conditions (A)–(C), then domain $D \neq \{\emptyset\}$ and the Bose condensate undergo a jump on the boundary ∂D:

$$\rho_0^B(\theta^{-1}, \mu) = \left\{ \begin{array}{c} > 0, (\theta, \mu) \in D \\ = 0, (\theta, \mu) \in Q \backslash \overline{D} \end{array} \right\} \tag{4.36}$$

where by definition: $\rho_0^B(\theta^{-1}, \mu = 0) \equiv \lim_{\mu \to 0^-} \rho_0^B(\theta^{-1}, \mu)$ (extension by continuity).

Behaviour of the trail pressure (4.26) and the condensate (4.36) are illustrated by figures 2 and 3.

5. Concluding remarks

This paper has presented an exact solution of the Bogoliubov WIBG model (1.4) originally conceived as a starting point for the explanation of superfluity [1, 2].

(i) We have shown that the thermodynamic properties of the model drastically depend on the interaction potential. We found that it is *non-diagonal part of interaction* that makes the model non-trivial (i.e. non-equivalent to the IBG)—theorems 2.16 and 3.14.

Therefore, we have answered the question formulated in [5] by showing that its solution depends on the potential. In particular we established that condition (C) (2.33) is necessary and sufficient for the WIBG be *non-equivalent* to the IBG in the domain of stability $Q = \{\theta \geqslant 0\} \times \{\mu \leqslant 0\}$.

(ii) We have shown that the Bogoliubov approximation for the WIBG is *exact* in the sense of theorem 3.14. It enables explicit calculation of the pressure $p^B(\beta, \mu)$. On the other hand this exact solution is rather different from the result of the Bogoliubov treatment [1, 2] of the Hamiltonian (1.4). This is because it involves additional hypotheses which are equivalent to modifications of the original Bogoliubov Hamiltonian (1.4) (see [10, 11, 16, 17] and references therein).

(iii) We have found that for interactions satisfying conditions (A)–(C) there is a domain (figure 3)

$$D = \{(\theta, \mu) : \mu_0 < \mu \leqslant 0, 0 \leqslant \theta < \theta_0(\mu)\} \subset Q$$

where the pressure $p_\Lambda^B(\beta, \mu) \neq p^I(\beta, \mu)$. We have shown (theorem 2.20) that in fact

$$D = \{(\theta, \mu) : \rho_0^B(\beta, \mu) \neq 0\}$$

where $\rho_0^B(\beta, \mu)$ is the density of the $k = 0$ mode Bose condensate in the Bogoliubov model.

(iv) It was shown (theorem 4.1) that the gauge symmetry is broken in D and that ρ_0^B changes its value on ∂D from $\rho_0^B = 0$ to $\rho_0^B \neq 0$ discontinuously (theorem 4.9).

(v) Moreover, the Hamiltonian $H_\Lambda^B(\hat{c}_\Lambda^\#, \mu)$, which is the thermodynamic equivalent to $H_\Lambda^B(\mu)$ (corollary 3.16), has a *gap* in the spectrum for $\lim_{k\to 0} E_k$ in domain D, i.e. in the presence of the Bose condensate (see (4.27)). This again indicates that the original Bogoliubov Hamiltonian (1.4) has been highly modified [1, 2, 5, 10, 11, 16] since its original invention for the description of superfluidity. The physical reason of the difference between the *exact solution* of the model H_Λ^B and the Bogoliubov theory [1, 2] is in the different treatment of *quantum fluctuations*.

It is the quantum fluctuations of the operators $a_0^\#/\sqrt{V}$ that imply an *effective attraction* between bosons with $k = 0$ in WIBG [12]. This attraction is the cause of two phenomena: instability of the WIBG for $\mu > 0$ (proposition 1.2) known since [5], and a *non-conventional* condensation of bosons in the $k = 0$ mode for negative μ when the *effective attraction* between them *dominates* a direct repulsion in (1.6) (see condition (C) (2.33) (or (24) in [12]) and theorems 2.16 and 2.20). By contrast, the Bogoliubov treatment of his model H_Λ^B (1.4) was based on the appoximation $a_0^\#/\sqrt{V} \to c^\#$ (2.24), i.e. on the *elimination* of the quantum fluctuations which makes the Hamiltonian $H_\Lambda^B(c^\#, \mu)$ (2.27) *stable* for a larger chemical potential domain: $\mu \leqslant v(0)|c|^2$ (i.e. even for $0 < \mu \leqslant v(0)|c|^2$ where the model (1.4) *does not exist*!). To make this treatment *self-consistent* and to gain a well known *gapless spectrum*, Bogoliubov's judicious choice of the parameter $|c|^2$ comes from the maximization of only the *non-fluctuating* 'Landau's part' of the trial pressure (2.28), i.e. of $\mu x - \frac{1}{2}v(0)x^2$ (for discussions see [16–18]). This choice bolsters the assertion of elimination of the quantum fluctuations for the Bogoliubov theory of superfluidity, but at the same time creates great debate about the role of the quantum fluctuations in the full Hamiltonian (1.4) in the presence of the condensate (as with the Hugenholtz–Pines theorem and Gavoret–Nozières analysis [19]) as well as mathematical papers about different model Hamiltonians with diagonal [20] and *non-diagonal* boson interactions [10, 11, 21] containing rigorous results on the Bose condensation in these interacting systems.

We have given the exact solution of the simplest non-diagonal model H_Λ^B (1.4) invented by Bogoliubov for WIBG. Instead of Bogoliubov treatment we considered the model H_Λ^B of WIBG rigorously, without any *a priori* ansatz or approximations. Our results (i)–(v) show that quantum fluctuations of operators $a_0^\#/\sqrt{V}$ make the properties of WIBG drastically different from the Bogoliubov treatment. This evidently means that the Bogoliubov theory of WIBG is something *more* that a simple study of the model $H_\Lambda^B(c^\#)$.

For example, our rigorous study of WIBG shows that the Bose condensate implies a gap in the excitation spectrum (see (v)) in contrast to the *aim* of the Bogoliubov theory. In fact, the nature of this gap is well known. The interaction in the *truncated* Hamiltonian H_Λ^B (in contrast to (1.2)) is *non-local*. This violates local gauge invariance and as a consequence the Hugenholtz–Pines–Gavoret–Nozières analysis (see an instructive discussion in [18] and the literature quoted there).

It may seem a *paradox* (cf the above remark about fluctuations) that the Bogoliubov approximation is *exact* (see (ii)) for calculations of the thermodynamic properties of the WIBG. In fact the quantum fluctuations (e.g. in the theorem 3.14) are not forgotten. They are responsible for the definition of domain D where the model H_Λ^B is stable and they define (in addition to the 'Landau part') a non-trivial 'fluctuating part' of the trial pressure (2.28).

Notice that the Bose condensation $\rho_0^B(\beta,\mu)$ in the WIBG for $\mu \leqslant 0$ (see (iii)) is due to effective attraction of the bosons in the mode $k = 0$ (see condition (C), (2.33) and theorem 2.20). We call it *non-conventional* (or *dynamical condensation*) in contrast to the *conventional* Bose condensation which is due to a simple *saturation* of occupation numbers in modes $k \neq 0$ [12]. The above study was done in the grand canonical ensemble by fixing temperature θ and chemical potential μ. Since the particle density $\rho^B(\beta,\mu) = \partial_\mu p^B(\beta,\mu)$ is bounded for $\mu \to 0^-$ (see (3.25), (3.26) and theorem 3.14), for densities $\rho > \rho^B(\beta, 0)$ one has to anticipate a *conventional* Bose condensation due to saturation of the density ρ^B. For $\theta < \theta_0\,(\mu = 0)$ this *conventional* condensation occurs *after* the *non-conventional condensation* $\rho_0^B(\beta,\mu)$. We return to these two scenarios of condensation elsewhere.

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Appendix A. Proof of theorem 3.7

(1) Notice that the pressure $p_\Lambda\left[H_\Lambda^B(\nu^{\#})\right]$ is bounded from below and from above uniformly on any compact $K \subset Q \times \{\nu : |\nu| < r_0\}$ see (2.25) and (3.20). Since the family

$$\{p_\Lambda[H_\Lambda^B(\nu^{\#})] = p_\Lambda^B(\beta,\mu;\nu^{\#})\}_{\Lambda\subset\mathbb{R}^3}$$

consists of convex functions of the chemical potential $\mu < 0$, by compactness argument (see e.g. ch II, section 10, [22]) there is a subsequence $\{p_{\Lambda_j}^B(\beta,\mu;\nu^{\#})\}_{j=1}^\infty$ which converges uniformly in μ on any compact $C_\mu \subset \mathbb{R}_-^1$ and fixed β, ν to the convex function $p^B(\beta,\mu;\nu^{\#})$, i.e. converges locally uniformly in $\mathbb{R}_-^1$.

(2) The grand-canonical pressure has the form:

$$p_\Lambda^B(\beta,\mu;\nu^{\#}) = \frac{1}{\beta V}\ln\left\{\sum_{N=0}^{\infty} \mathrm{e}^{\beta V(\mu\frac{N}{V} - f_\Lambda^B(\beta,\frac{N}{V};\nu^{\#}))}\right\} = (\beta V)^{-1}\ln \Xi_\Lambda^B(\beta,\mu;\nu^{\#}) \tag{A.1}$$

where

$$f_\Lambda^B(\beta,\rho = N/V;\nu^{\#}) = -\frac{1}{\beta V}\,\mathrm{Tr}_{\mathcal{F}_N}\,\mathrm{e}^{-\beta H_\Lambda^B(\nu^{\#})} \qquad \rho \geqslant 0 \tag{A.2}$$

is the free-energy density. By conditions of the theorem 3.7 $(\theta,\mu) \in D$, which corresponds to the *one-phase domain* : $\rho_0^B > 0$. Consequently, $\{\partial_\mu p_\Lambda^B = \langle\frac{N}{V}\rangle_{H_\Lambda^B(\nu^{\#})} > 0\}_\Lambda$ and $\partial_\mu p^B =$ (the Griffiths lemma) $= \lim_\Lambda \partial_\mu p_\Lambda^B$ are continuous functions of $\mu \in [\mu_0(\beta)+\varepsilon, 0)$, $\varepsilon > 0$. The $\partial_\mu p^B$ can be extended to $\mu = 0$ by continuity. Then by a Tauberian theorem proved in [23] the existence of the limit $p^B(\beta,\mu;\nu^{\#})$ entails the existence of the limit

$$f^B(\beta,\rho;\nu^{\#}) = \lim_\Lambda f_\Lambda^B(\beta,\rho;\nu^{\#}) \tag{A.3}$$

which is uniform on the interval

$$\rho \in I_{\varepsilon_{1,2}} = [\partial_\mu p^B(\beta, \mu_0(\beta) + \varepsilon_1; \nu^\#), \partial_\mu p^B(\beta, -\varepsilon_2; \nu^\#)] \qquad \varepsilon_{1,2} > 0. \tag{A.4}$$

In fact on this interval the limit (A.3) coincides with its convex envelope (the *Legendre transformation*):

$$\tilde{f}^B(\beta, \rho; \nu^\#) = \text{C.E.}\ \{f^B(\beta, \rho; \nu^\#)\} = \sup_{\mu\leqslant 0}\{\mu\rho - p^B(\beta, \mu; \nu^\#)\}. \tag{A.5}$$

(3) By virtue of (A.3) and (A.5), for $|\Lambda_j|$ large enough functions $\{f^B_{\Lambda_j}(\beta, \rho; \nu^\#)\}_{j=1}^\infty$ are strictly convex on $I_{\varepsilon_{1,2}}$ and for $\mu \in [\mu_0(\beta) + \varepsilon_1, 0]$

$$\sup_{\frac{N}{V}} \left(\mu\frac{N}{V} - f^B_\Lambda\left(\beta, \frac{N}{V}; \nu^\#\right)\right) = \mu\overline{\rho}_\Lambda - f^B_\Lambda(\beta, \overline{\rho}_\Lambda; \nu^\#) \equiv -F_\Lambda(\beta, \mu; \overline{\rho}_\Lambda, \nu^\#) \tag{A.6}$$

$\overline{\rho}_\Lambda(\mu) \in I_{\varepsilon_{1,2}}$. Then for $|\frac{N}{V} - \overline{\rho}_\Lambda| > \xi > 0$ one gets

$$F_\Lambda\left(\beta, \mu; \frac{N}{V}, \nu^\#\right) > F_\Lambda(\beta, \mu; \overline{\rho}_\Lambda, \nu^\#) + \gamma \equiv \overline{F}_\Lambda + \gamma \qquad \gamma > 0 \tag{A.7}$$

and for $|\frac{N}{V} - \overline{\rho}_\Lambda| < \xi' < \xi$ one has

$$\overline{F}_\Lambda \leqslant F_\Lambda\left(\beta, \mu; \frac{N}{V}, \nu^\#\right) \leqslant \overline{F}_\Lambda + \frac{\gamma}{2}. \tag{A.8}$$

By (A.1) one gets that for $\frac{N}{V} - \overline{\rho}_\Lambda > \xi$ there are two constants $a_{1,2} > 0$ such that

$$F_\Lambda\left(\beta, \mu; \frac{N}{V}, \nu^\#\right) > a_1 + a_2\left(\frac{N}{V} - \overline{\rho}_\Lambda - \xi\right). \tag{A.9}$$

(4) (*Large-deviation principle for the particle density*). By standard reasoning (see e.g. [24, 25]) one gets from the grand-canonical distribution of particles and (A.7)–(A.9) that

$$p_{\Lambda,I} = \mathbb{P}_\Lambda\left\{0 \leqslant \frac{N}{V} < \overline{\rho}_\Lambda - \xi\right\} = (\Xi^B_\Lambda)^{-1} \sum_{0\leqslant N < V(\overline{\rho}_\Lambda - \xi)} \mathrm{e}^{-\beta V F_\Lambda(\beta,\mu;\frac{N}{V},\nu^\#)}$$
$$\leqslant V(\overline{\rho}_\Lambda - \xi)\mathrm{e}^{-\beta V(\overline{F}_\Lambda + \gamma)} \tag{A.10}$$

$$p_{\Lambda,II} = \mathbb{P}_\Lambda\left\{\overline{\rho}_\Lambda - \xi \leqslant \frac{N}{V} < \overline{\rho}_\Lambda + \xi\right\} \geqslant (\Xi^B_\Lambda)^{-1} \sum_{V(\overline{\rho}_\Lambda - \xi')\leqslant N < V(\overline{\rho}_\Lambda + \xi')} \mathrm{e}^{-\beta V F_\Lambda(\beta,\mu;\frac{N}{V},\nu^\#)}$$
$$\geqslant 2(\Xi^B_\Lambda)^{-1}\xi' V e^{-\beta V(\overline{F}_\Lambda + \frac{\gamma}{2})} \tag{A.11}$$

$$p_{\Lambda,III} = \mathbb{P}_\Lambda\left\{\frac{N}{V} \geqslant \overline{\rho}_\Lambda + \xi\right\} \leqslant (\Xi^B_\Lambda)^{-1} \sum_{N\geqslant V(\overline{\rho}_\Lambda + \xi)} \mathrm{e}^{-\beta V[a_1 + a_2(\frac{N}{V} - \overline{\rho}_\Lambda - \xi)]}. \tag{A.12}$$

Since $p_{\Lambda,I} + p_{\Lambda,II} + p_{\Lambda,III} = 1$, these estimates imply that

$$\lim_\Lambda p_{\Lambda,II} = 1 \tag{A.13}$$

for any $\xi > 0$.

(5) Now we can apply the large-deviation principle for the particle density in domain D to obtain (3.19). Using the relation (3.35) for $A^\#$ one readily gets

$$\langle A^*[A, [H^B_\Lambda(\mu, \nu^\#), A^*]]A\rangle_{H^B_\Lambda(\nu^\#)} \leqslant 2v(0)\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda(\nu^\#)} - \mu\langle A^*A\rangle_{H^B_\Lambda(\nu^\#)}. \tag{A.14}$$

Since

$$\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda(\nu^\#)} = \left\langle\left(\frac{N_\Lambda}{V} - \overline{\rho}_\Lambda\right)A^*A\right\rangle_{H^B_\Lambda(\nu^\#)} + \overline{\rho}_\Lambda\langle A^*A\rangle_{H^B_\Lambda(\nu^\#)} \tag{A.15}$$

we have to estimate from above the first term in the left-hand side of (A.15). To this end we follow (A.10)–(A.13):

(I)

$$(\Xi_\Lambda^B)^{-1}\sum_{0\leqslant N<V(\overline{\rho}_\Lambda-\xi)}\left(\frac{N}{V}-\overline{\rho}_\Lambda\right)\mathrm{e}^{-\beta VF_\Lambda}\langle A^*A\rangle_{H_\Lambda^B(\nu^\#)}(\beta,N;\nu^\#)\leqslant 0. \quad (A.16)$$

(II)

$$(\Xi_\Lambda^B)^{-1}\sum_{V(\overline{\rho}_\Lambda-\xi)\leqslant N<V(\overline{\rho}_\Lambda+\xi)}\mathrm{e}^{\beta\mu N}\left(\frac{N}{V}-\overline{\rho}_\Lambda\right)\mathrm{Tr}_{\mathcal{F}_N}(\mathrm{e}^{-\beta H_\Lambda^B(\nu^\#)}A^*A)$$
$$\leqslant 2\xi(\Xi_\Lambda^B)^{-1}\sum_{N=0}^{\infty}\mathrm{e}^{\beta\mu N}\,\mathrm{Tr}_{\mathcal{F}_N}(\mathrm{e}^{-\beta H_\Lambda^B(\nu^\#)}A^*A)=\xi\langle A^*A\rangle_{H_\Lambda^B(\nu^\#)}(\beta,\mu;\nu^\#). \quad (A.17)$$

(III)

$$\begin{aligned}(\Xi_\Lambda^B)^{-1}&\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}\left(\frac{N}{V}-\overline{\rho}_\Lambda\right)\mathrm{e}^{-\beta VF_\Lambda}\langle A^*A\rangle_{H_\Lambda^B(\nu^\#)}(\beta,N;\nu^\#)\\
&\leqslant(\Xi_\Lambda^B)^{-1}\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}\left(\frac{N}{V}-\overline{\rho}_\Lambda\right)\mathrm{e}^{-\beta V[a_1+a_2(\frac{N}{V}-\overline{\rho}_\Lambda-\xi)]}2(N+|c|^2V)\\
&=2(\Xi_\Lambda^B)^{-1}\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}\left(\frac{N}{V}-\overline{\rho}_\Lambda\right)(N-V\overline{\rho}_\Lambda)\mathrm{e}^{-\beta Va_1}\mathrm{e}^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))}\\
&\quad+2(\Xi_\Lambda^B)^{-1}V(\overline{\rho}_\Lambda+|c|^2)\mathrm{e}^{-\beta Va_1}\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}\mathrm{e}^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))}\\
&=p_{\Lambda,II}(\xi'V^2)^{-1}\mathrm{e}^{-\beta V\frac{\gamma}{2}}\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}(N-V\overline{\rho}_\Lambda)^2\mathrm{e}^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))}\\
&\quad+p_{\Lambda,II}V^{-1}(\overline{\rho}_\Lambda+|c|^2)\mathrm{e}^{-\beta V\frac{\gamma}{2}}\sum_{N\geqslant V(\overline{\rho}_\Lambda+\xi)}\mathrm{e}^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))}\leqslant\text{constant }\mathrm{e}^{-\beta V\frac{\gamma}{2}}.\end{aligned}$$

(A.18)

Combining (A.16)–(A.18) with (A.14) and (A.15) we find that for any compact $C_\mu\subset(\mu_0(\beta),0)$, $|\nu|\leqslant r_0$ and compact $C_\beta\subset\mathbb{R}^1_+$ one has

$$\langle A^*[A,[H_\Lambda^B(\mu,\nu^\#),A^*]]A\rangle_{H_\Lambda^B(\nu^\#)}\leqslant a\langle A^*A\rangle_{H_\Lambda^B(\nu^\#)}+b \quad (A.19)$$

for positive bounded a, b which depend on C_μ, C_β, and r_0, i.e. for a, b locally bounded in D.

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